Online appendix to“Corporate Debt Structure and the Macroeconomy”

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Preliminary; latest version at: `www.columbia.edu/~nc2371/research/JMP_appendix.pdf`
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This appendix contains results, definitions and proofs omitted from the main text of my job market paper, “Corporate Debt Structure and the Macroeconomy”. Equation or figure xx from the main text is referenced as **(M-xx)**.

1 Data appendix

Data on aggregate financial ratios For the US, the data on the aggregate bank share is obtained from table L.102 of the Flow of Funds, the balance sheet of the the nonfinancial corporate business sector. The series in the left panel of figure **(M-13)** is the ratio of the sum of depository institution loans (line 27) and other loans and advances (line 28) to total credit market instruments oustanding (line 23). The ratio of debt to assets is measured as the ratio of total credit market instruments outstanding (line 23), to miscellaneous assets (line 16), a measure of assets excluding credit market instruments and deposits or money market fund shares. I exclude these financial assets from this ratio because the model's firms do not hold cash and do not lend to other firms.

For Italy, the data on the aggregate bank share is obtained from Bank of Italy (2008), table 5 (TD-HET000). The aggregate bank share is computed as the ratio of total loans (short and long-term) to total debt. Total debt is measured as total liabilities minus shares and other equities issued by residents.[1] Similarly to the US, I construct the ratio of debt to assets as the ratio of total debt to a measure of total assets which excludes cash and cash-like securities, as well as credit market securities. I obtain this measure by substracting deposits, short-terms securities, bonds, derivatives, short-term loans and mutual fund shares from total liabilities of firms.

Figure **(M-10)** reports aggregate bank shares for a larger sample of countries, on average, between 2000 and 2007. This graph is obtained using firm-level data from the OSIRIS database. For each year

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[1] The Bank of Italy distinguishes between loans from MFI's (monetary financial institutions, comprising the Central Bank, banks, money-market funds, electronic money institutions and the Cassa Depositi e Prestiti), and loans from other financial institutions. Excluding other financial institutions, the aggregate bank share for 2007Q3 is 50.3 %, closer to the aggregate bank share which can be matched by the model.

t between 2000 and 2007, each country c, and each firm j in that country's set of firms $F_{c,t}$, I construct measures of outstanding bank debt and outstanding total debt for each firm, $b^c_{j,t}$ and $d^c_{j,t}$. The construction of these measures is described below. For each country-year, the aggregate bank share is computed as $BS_{c,t} = \frac{\sum_{j \in F_{c,t}} b_{j,t}}{\sum_{j \in F_{c,t}} d_{j,t}}$. The aggregate bank shares reported in the graph are the average of the $BS_{c,t}$ for $t = 2000, ..., 2007$.[2]

Data on business-cycle changes in debt composition in the US The middle and right panels of figure **(M-13)** report changes in outstanding bank and non-bank credit for small and large manufacturing firms, in the US, from 2008 onwards. The data is from the Quarterly Financial Report of manufacturing firms.[3] This dataset contains information on firms' balance sheets and income statements, and is reported in semi-aggregated form (by asset size categories). The QFR has two advantages over firm-level datasets: it includes small and private firms as well as large firms; and it has quarterly coverage. By contrast, the other firm-level dataset I use for the US, created by Rauh and Sufi (2010), covers only public firms, is annual, and does not extend to 2012. It is thus less adapted to documenting facts on business-cycle changes in debt composition.

Crouzet (2013), section 2 and appendix D, discusses in detail the QFR and the variables definition used. The "small firm" category is defined as firms with less than $1bn\$$ in assets, and the "large firm" category as the remainder.[4] For both categories, total debt is defined as total liabilities excluding non-financial liabilities (such as trade credit) and stockholders' equity. It includes both short and long-term debt. Bank debt b_t is reported as a specific item in the QFR; I define market credit m_t as the remaining financial liabilities. The series denoted "bank debt" in figure **(M-13)** is given by: $\gamma_{b,small,t} = \frac{b_{small,t_0}}{b_{small,t_0}+m_{small,t_0}} \left(\frac{b_{small,t}}{b_{small,t_0}} - 1 \right)$. The series denotes "market debt" is defined similarly. This preserves additivity; that is, the sum of the two lines in figure **(M-13)** corresponds to the percent change in total financial liabilities around t_0. I choose t_0 to be $2008Q3$. This is the date of the failure of Lehman Brothers, and it also marks the start in the decline of the aggregate bank share (see left panel). The series reported in this figure are smoothed by a 2 by 4 MA smoother to remove seasonal variation.

Data on firm-level debt composition Figure **(M-8)** is constructed using firm-level data from the OSIRIS database maintained by Bureau Van Djik. This dataset contains balance sheet information for publicly traded firms in emerging and advanced economies. I focus on the subsample of non-financial firms

[2]In particular, the aggregate bank shares obtained in this way for Italy and the US are not the same as those obtained using the financial accounts of either country. I choose to use firm-level data in the construction of figure **(M-10)** for those countries, rather than the financial accounts data, to maintain comparability with other countries.

[3]See `http://www.census.gov/econ/qfr/`.

[4]Crouzet (2013) shows that the pattern of debt substitution is robust to different definitions of size groups, in particular that adopted by Gertler and Gilchrist (1994).

that are active between 2000 and 2010, and keep only firms that report consolidated financial statements.

Total debt $d^c_{j,t}$ is defined as total long-term interest bearing debt (data item number 14016), and bank debt $b^c_{j,t}$ as bank loans (data item number 21070). I focus on long-term debt (inclusive of the currently due portions of it) because bank loans are a subset of this category in the OSIRIS database; currently due debt features a "loans" category which include potentially credit instruments other than bank loans. Additionally, equity $e^c_{j,t}$ is defined as shareholders funds (data item 14041). For each firm-year observation, the bank share is defined as: $s_{j,t,c} = \frac{b^c_{j,t}}{d^c_{j,t}}$. I keep only observations for which the $e_{j,t,c} \geq 0$ and $s_{j,t,c} \in [0,1]$ (and for which both are non-missing). The resulting sample has 51921 firm-year observations, correspondind to 12931 distinct company names.

Observations are pooled by country-year (t,c). For each country-year cell, let $\tilde{e}_{k,t,c}$ denote the $k-th$ quantile of the empirical distribution of firms across equity levels $e_{j,t,c}$. I use $k \in \{5, 15, 25, ..., 95\}$. Define the *average* bank share within each quantile group as:

$$\tilde{s}_{k_i,t,c} = \frac{1}{N_{k_i,t,c}} \sum_{\tilde{e}_{k_i,t,c} \leq e_{j,t,c} < \tilde{e}_{k_{i+1},t,c}} s_{j,t,c},$$

where $N_{k_i,t,c}$ is the number of firms in (t,c) with $\tilde{e}_{k_{i-1},t,c} \leq e_{j,t,c} < \tilde{e}_{k_i,t,c}$.[5] I then average out these shares over time: $\tilde{s}_{k_i,c} = \frac{1}{T}\sum_t \tilde{s}_{k_i,t,c}$. Figure **(M-8)** reports the pairs $(k_i, \tilde{s}_{k_i,c})$, for the subset of 8 countries that have the largest number of observations among advanced and emerging economies, respectively.

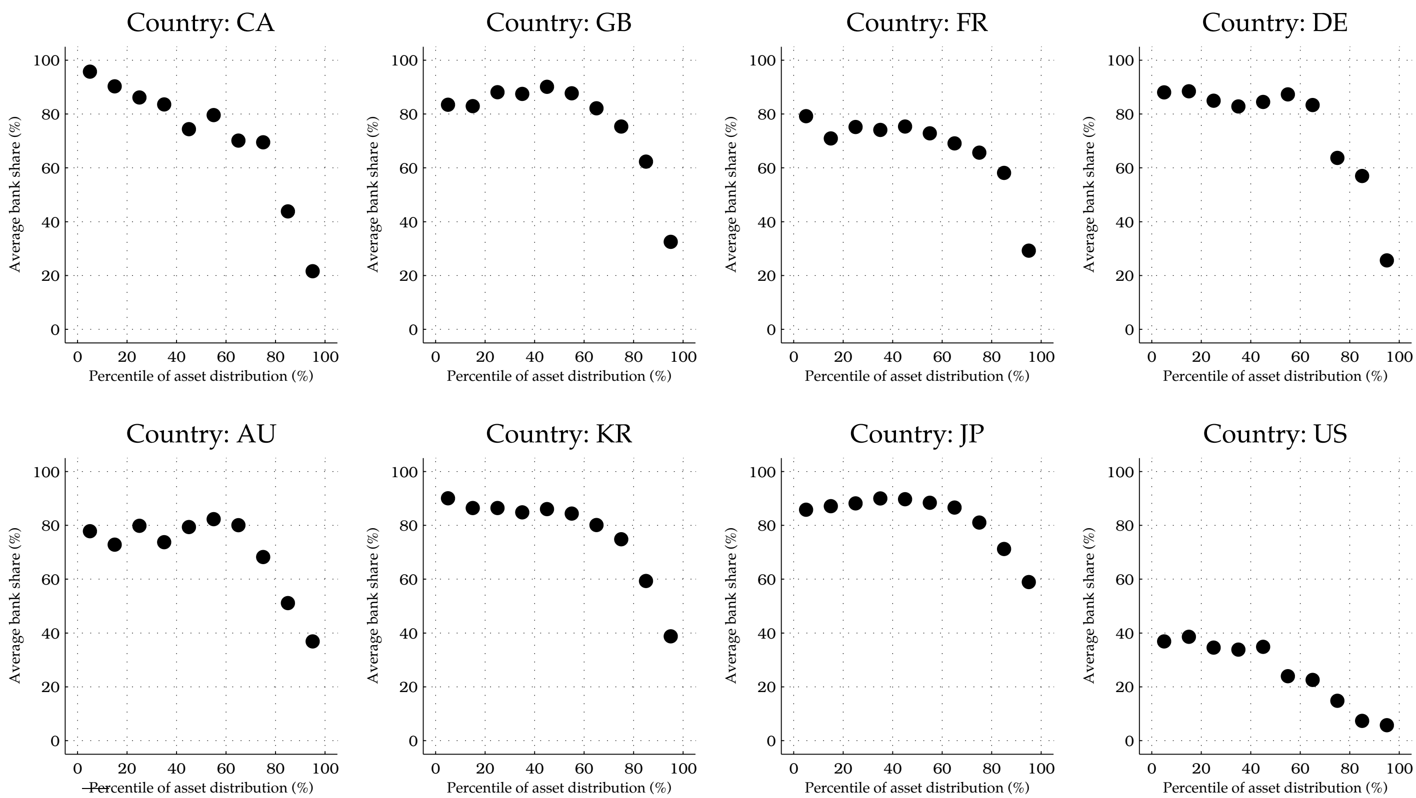
I do this for all countries, except for the US. For the US, my firm-level data is drawn from the dataset created by Rauh and Sufi (2010), arguably the highest-quality dataset on debt structure for publicly traded-firms. This dataset draws from Compustat (for balance-sheet data) and Dealscan (for bond issuances), and covers a larger number of firms than those available for the US wihtin the OSIRIS database, for a similar period. This database has direct measures of total debt $d_{j,t,US}$. Equity $e_{j,t,US}$ is defined as the difference between the "debt plus equity" variable and the "debt" variable. I keep only firms-year observations for which this measure of equity is positive. Finally, bank debt $b_{j,t,US}$ is defined as outstanding bank loans. All other variable definitions are identical.[6]

The relationship between total assets and the bank share A natural alternative measure of firm size are the value of its assets. In the cross-section, asset value is negatively related to the amount of bank in a firms' debt structure. I document this using the same cross-sectional statistics as for net worth described above, that is, the average bank share by percentile of the asset size distribution. For all countries in the

[5] Clearly, for $k_i = 5\%$, $k_{i-1} = 0$ and for $k_{i-1} = 95\%$, $k_i = 100\%$.

[6] In the US sample of the OSIRIS database, the negative relationship between bank share $\widetilde{k_i, US}$ and equity size k_i also obtains, but the average level of the bank share is higher, and in fact larger than in other advanced countries.

(a) **Total assets and bank share in OECD-countries**



(b) **Total assets and bank share in non-OECD-countries**

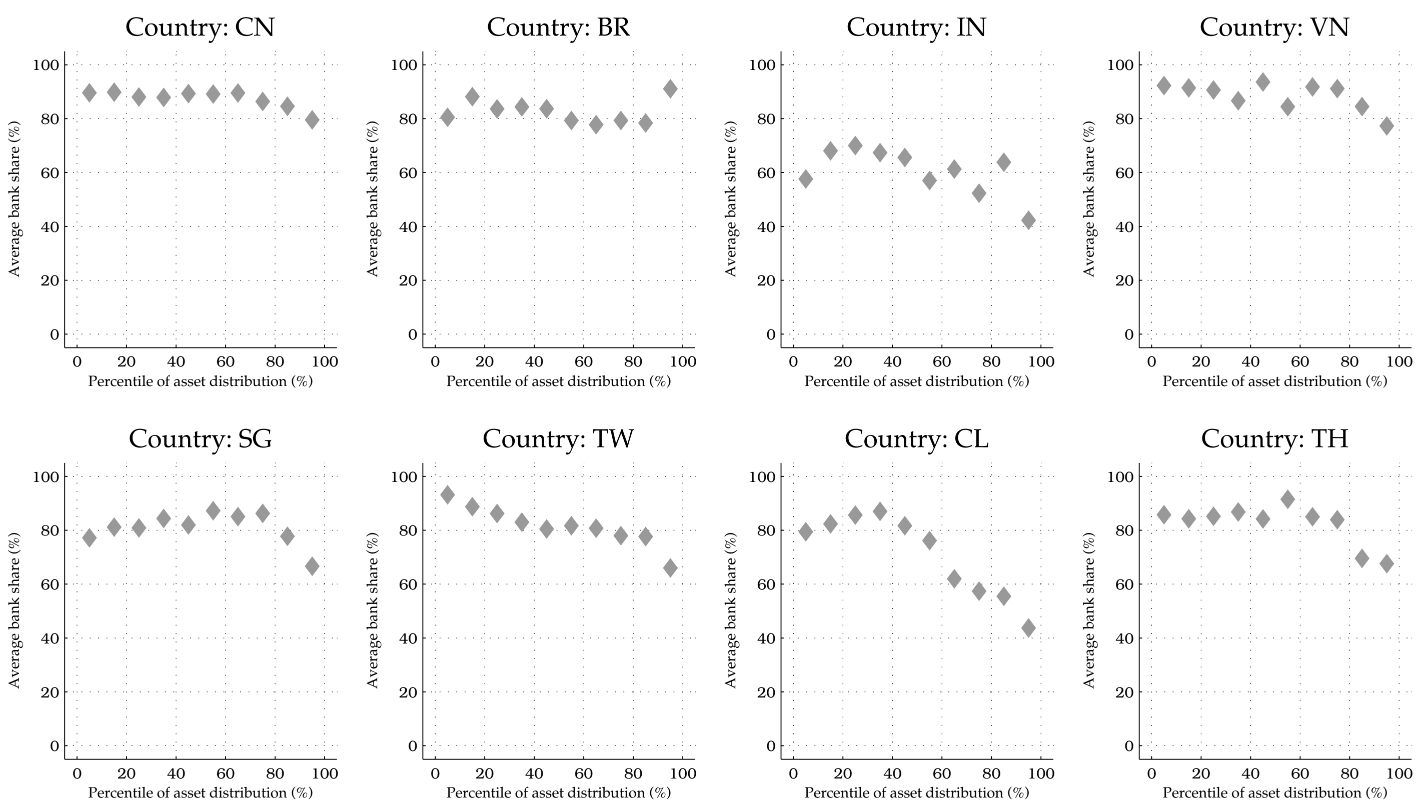


Figure 1: Bank share and fixed assets in the cross-section. Each graph reports, for a particualr country, the median ratio of bank loans to total firm liabilities, in each decile of the asset distribution. For the US, data from taken from Rauh and Sufi (2010); for other countries, data from Bureau Van Djik.

OSIRIS database, asset value is the book value of fixed assets (data item 20085) which includes property, plant, equipment, intangibles, and other fixed assets. For the US, I use the measure of assets provided by Rauh and Sufi (2010). Fixed assets is the more directly relevant measure of assets since firms in the model only hold real assets; however, using total assets (inclusive of financial assets, a measure also available in the OSIRIS database) does not change the results. The negative relationship between this measure of size and the bank share is reported in figure 1.

2 Analytical proofs

Proof of lemma 1.

When V is continuous, the objective function in problem **(M-1)** is continuous. The constraint correspondence in **(M-1)** is compact-valued and continuous. The theorem of the maximum then implies that V^c is continuous. Let $\left(n_t^1, n_t^2\right) \in \mathbb{R}_+^2$ such that $n_t^1 > n_t^2$, and let e_{t+1}^2 be a value for next period equity that solves **(M-1)**, when $n_t = n_t^2$. We have $e_{t+1}^2 \leq n_t^2 < n_t^1$, so e_{t+1}^2 is also feasible when $n_t = n_t^1$. Therefore, $V^c(n_t^1) \geq n_t^1 - e_{t+1}^2 + (1-\eta)\beta V(e_{t+1}^2) > n_t^2 - e_{t+1}^2 + (1-\eta)\beta V(e_{t+1}^2) = V^c(n_t^2)$. This proves that V^c is strictly increasing. Finally, when $n_t = 0$ the feasible set contains only $div_t = 0, e_{t+1} = 0$. So $V^c(0) = (1-\eta)\beta V(0)$. Therefore when $V(0) \geq 0$, $V^c(0) \geq 0$.

■

Proof of proposition 1. A useful result for the proof is that $V^c(n_t) \geq n_t$ when $V(0) \geq 0$. This is established by noting that the dividend policy $e_{t+1} = 0$ is always feasible at the dividend issuance stage, and that the value of this policy is $n_t + (1-\eta)\beta V(0) \geq n_t$.

Assume, first, that $\frac{R_{m,t}}{1-\chi} \leq \frac{R_{b,t}}{\chi}$. Then, $\frac{R_{m,t}}{1-\chi} \leq R_{b,t} + R_{m,t} \leq \frac{R_{b,t}}{\chi}$. The proof proceeds by comparing V_t^L, V_t^R and V_t^P, the values of the firm under liquidation, restructuring or repayment, for each realization of π_t. There are five possible cases:

- when $\pi_t \geq \frac{R_{b,t}+R_{m,t}}{\chi}$, we have $V_t^L = \chi\pi_t - R_{b,t} - R_{m,t} < \pi_t - R_{b,t} - R_{m,t} \leq V^c(\pi_t - R_{b,t} - R_{m,t}) = V_t^P$. Moreover, since $\pi_t \geq \frac{R_{b,t}+R_{m,t}}{\chi} \geq \frac{R_{b,t}}{\chi}$, the reservation value of the bank is $R_{b,t}$, so the best restructuring offer for the firm is $l_t = R_{b,t}$. Therefore $V_t^P = V_t^R$. I will assume the firm chooses repayment.

- when $\frac{R_{b,t}+R_{m,t}}{\chi} > \pi_t \geq \frac{R_{b,t}}{\chi}$, we have $V_t^L = 0 \leq V^c(\pi_t - R_{b,t} - R_{m,t}) = V_t^P$, since $\pi_t \geq \frac{R_{b,t}}{\chi} \geq R_{b,t} + R_{m,t}$, $V^c(0) \geq 0$ and V^c is strictly increasing. ($V_t^L < V_t^R$ so long as $\pi_t > R_{b,t} + R_{m,t}$). Moreover, $V_t^R = V_t^P$ for the same reason as above. Again, the firm chooses repayment.

- when $\frac{R_{b,t}}{\chi} > \pi_t \geq R_{b,t} + R_{m,t}$, the reservation value of the bank is $\chi\pi_t$. The restructuring offer at which the participation constraint of the bank binds, $l_t = \chi\pi_t$, is feasible because $\pi_t - l_t - R_{m,t} = (1-\chi)\pi_t - R_{m,t} \geq 0$. So $V_t^R \geq V^c((1-\chi)\pi_t - R_{m,t})$. This implies $V_t^R > V^c(\pi_t - R_{b,t} - R_{m,t}) = V_t^P$, since V^c is strictly increasing and $(1-\chi)\pi_t - R_{m,t} > \pi_t - R_{b,t} - R_{m,t}$. For the same reasons as above, $V_t^R \geq V_t^L$. So the firm chooses to restructure. Because V^c is increasing, the optimal restructuring offer makes the participation constraint of the bank bind: $\hat{l}_t = \chi\pi_t$.

- when $R_{b,t} + R_{m,t} \geq \pi_t \geq \frac{R_{m,t}}{1-\chi}$, we have $V_t^L = 0 < V^c((1-\chi)\pi_t - R_{m,t}) = V_t^R$, where again the properties of V^c were used. Moreover, $V_t^P = V_t^L$, since the firm does not have enough funds to repay both its creditors. So the firm chooses to restructure, again with $\hat{l}_t = \chi\pi_t$.

- when $\pi_t < \frac{R_{m,t}}{1-\chi}$, the firm is liquidated because any restructuring offer consistent with the participation constraint of the bank will leave the firm unable to repaye market creditors. Since in that case, $0 > \chi\pi_t - R_{b,t} - R_{m,t}$, the liquidation value for the firm is $V_t^L = 0$.

This shows that the repays when $\pi_t \geq \frac{R_{b,t}}{\chi}$, restructures when $\frac{R_{b,t}}{\chi} \geq \pi_t \geq \frac{R_{m,t}}{1-\chi}$, and is liquidated otherwise. Moreover, this also establishes, for this case, the two additional claims of the proposition: $V_t^L = 0$ whenever liquidation is chosen, and the restructuring offer always makes the participation constraint of the bank bind: $\hat{l}_t = \chi\pi_t$. The claims of the proposition when $\frac{R_{m,t}}{1-\chi} > \frac{R_{b,t}}{\chi}$ can similarly be established, by focusing on the three sub-cases $\pi_t \geq \frac{R_{b,t}+R_{m,t}}{\chi}$, $\frac{R_{b,t}+R_{m,t}}{\chi} > \pi_t \geq R_{b,t} + R_{m,t}$ and $R_{b,t} + R_{m,t} > \pi_t$.

■

Proof of proposition 2. Given proposition 1, the debt settlement outcomes yield the same conditional return functions for banks and market lenders, $\tilde{R}_{b,t}(\pi_t, R_{b,t}, R_{m,t})$ and $\tilde{R}_{m,t}(\pi_t, R_{b,t}, R_{m,t})$, as those derived in the static model of Crouzet (2013). Therefore, all the results of section 3.4 of that paper apply. Proposition 2 is a subset of the results of that papers' proposition 2. ■

Proof of proposition 3

The proof of the existence of a recursive competitive equilibrium in the economy of section 2 of the main paper contains two broad steps. First, one needs to prove that the optimal debt structure problem of a single firm, problem **(M-8)**, has a unique solution. Second, one must establish the existence and unicity of a steady-state distribution of firms across equity sizes. I start by introducing some notation.

Preliminary notation Throughout, I restate the firm's problem in terms of the variables $d_t = b_t + m_t$ and $s_t = \frac{b_t}{b_t+m_t}$. d_t denotes total borrowing by the firm, and s_t denotes the share of borrowing that is bank debt. Note that $(d_t, s_t) \in \mathbb{R}_+ \times [0,1]$. With some abuse of notation, I will keep denoting the set of feasible debt structures (d_t, s_t) by $\mathcal{S}(e_t)$, and its partition established in proposition 2 as $(\mathcal{S}_K(e_t), \mathcal{S}_R(e_t))$. Additionally, define the functions $G : \mathbb{R}_+ \to \mathbb{R}_+$, $I(.; e_t + d_t) : \mathbb{R}_+ \to \mathbb{R}_+$ and $M(; e_t + d_t) : \mathbb{R}_+ \to \mathbb{R}_+$ by:

$$\begin{aligned} G(x) &= x(1 - F(x)) + \int_0^x \phi dF(\phi) \\ I(x; e_t + d_t) &= x(1 - F(x)) - F(x)(1 - \delta)(e_t + d_t)^{1-\zeta} \\ M(x; e_t + d_t) &= (1 - \chi)I(x; e_t + d_t) + \chi G(x) \end{aligned}$$

Following the results of Crouzet (2013), lemmas 3 to 5, G is strictly increasing in R^+, while I and M have unique maxima. Moreover, the terms of debt contracts $(R_{b,t}, R_{m,t})$ for given (e_t, d_t, s_t) can be expressed using the inverse mappings of these three functions, denoted by G^{-1}, $I^{-1}(, ; e_t + d_t)$ and $M^{-1}(.; e_t + d_t)$. These inverse mappings are defined, respectively, on $[0, \mathbb{E}(\phi)]$, $\left[0, \hat{I}(e_t + d_t)\right]$ and $\left[0, \hat{M}(e_t + d_t)\right]$, where $\hat{I}(e_t + d_t)$ is the global maximum of I and similarly $\hat{M}(e_t + d_t)$ is the global maximum of M. For example, the terms of bank contracts when $(d_t, s_t) \in \mathcal{S}_R(e_t)$ are given by:

$$R_{b,t} = R_b(d_t, s_t, e_t) = \begin{cases} (1 + r_b)d_t s_t & \text{if} \quad 0 \le \frac{(1+r_b)d_t s_t}{\chi(e_t+d_t)^\zeta} < (1 - \delta)(e_t + d_t)^{1-\zeta} \\ \chi(1 - \delta)(e_t + d_t) & \text{if} \quad (1 - \delta)(e_t + d_t)^{1-\zeta} \le \frac{(1+r_b)d_t s_t}{\chi(e_t+d_t)^\zeta} \\ \qquad +\chi(e_t + d_t)^\zeta G^{-1}\left(\frac{(1+r_b)d_t s_t - \chi(1-\delta)(e_t+d_t)}{\chi(e_t+d_t)^\zeta}\right) & \qquad \le \mathbb{E}(\phi) + (1 - \delta)(e_t + d_t)^{1-\zeta} \end{cases}$$

The expressions for $R_m(d_t, s_t, e_t)$ when $(d_t, s_t) \in \mathcal{S}_R(e_t)$ and for $R_b(d_t, s_t, e_t)$ and $R_m(d_t, s_t, e_t)$ when $(d_t, s_t) \in \mathcal{S}_K(e_t)$ are reported in Crouzet (2013). In what follows, I use these results to express explicitly the thresholds for restructuring and liquidation obtained in proposition 1, namely:

$$\begin{aligned} \underline{\phi}_R(e_t, d_t, s_t) &= \frac{R_m(d_t,s_t,e_t) - (1-\chi)(1-\delta)(e_t+d_t)}{(1-\chi)(e_t+d_t)^\zeta} && \text{(liquidation threshold when } (d_t, s_t) \in \mathcal{S}_R(e_t)) \\ \overline{\phi}_R(e_t, d_t, s_t) &= \frac{R_b(d_t,s_t,e_t) - \chi(1-\delta)(e_t+d_t)}{\chi(e_t+d_t)^\zeta} && \text{(restructuring threshold when } (d_t, s_t) \in \mathcal{S}_R(e_t)) \\ \underline{\phi}_K(e_t, d_t, s_t) &= \frac{R_b(d_t,s_t,e_t) + R_m(d_t,s_t,e_t) - (1-\delta)(e_t+d_t)}{(e_t+d_t)^\zeta} && \text{(liquidation threshold when } (d_t, s_t) \in \mathcal{S}_K(e_t)) \end{aligned}$$

Proof of existence and unicity of a solution to problem (M-8). The proof has three steps:

Step 1: Reformulate the optimization problem of the firm as the combination of a discrete choice and continuous choice problem.

Step 2: Show that the functional mapping T associated with this new formulation maps the space $C(E)$ of real-valued, continuous functions on $[0, E]$, with the sup norm $\|.\|_s$, onto itself, where $E > 0$ is an arbitrarily large upper bound for equity. Additionally, show that $T(C_0(E)) \subseteq C_0(E)$, where $C_0(E) = \{V \in C(E) \text{ s.t. } V(0) \ge 0\}$. Since $(C(E), \|.\|_s)$ is a complete metric space and $C_0(E)$ is a closed subset of $C(E)$ under $\|.\|_s$ which is additionally stable through T, if T is a contraction mapping, then its fixed point must be in $C_0(E)$.

Step 3: Check that T satisfies Blackwell's sufficiency conditions, so that it is indeed a contraction mapping.

Note that step 2 is crucial because lemma 1 requires the continuity of V and the fact that $V(0) \geq 0$ for V^c to be continuous, strictly increasing and satisfy $V^c(0) \geq 0$. In turn, these three conditions are necessary for characterizing of the set of feasible debt structures, that is, for propositions 1 and 2 to hold.

Step 1: Define the mapping T on $C(E)$ as:

$$\forall e_t \in [0, E], \quad TV(e_t) = \max_{R,K} \Big(T_R V(e_t), T_K V(e_t) \Big) \tag{A1}$$

where the mappings T_R and T_K, also defined on $C(E)$, are given by:

$$\forall e_t \in [0, E], \quad T_R V(e_t) \quad = \quad \max_{(d_t, s_t) \in \mathcal{S}_R(e_t)} \int_{\phi_t \geq \underline{\phi}_R(e_t, d_t, s_t)} V^c \left(n_R \left(\phi_t; e_t, d_t, s_t \right) \right) dF(\phi_t) \tag{A1-R}$$

$$\text{s.t.} \quad V^c(n_t) = \max_{0 \leq e_{t+1} \leq n_t} n_t - e_{t+1} + (1-\eta)\beta V(e_{t+1})$$

$$n_R(\phi_t, e_t, d_t, s_t) = \begin{cases} \Big(\phi_t - \chi \overline{\phi}_R(e_t, d_t, s_t) - (1-\chi) \underline{\phi}_R(e_t, d_t, s_t) \Big) (e_t + d_t)^\zeta & \text{if} \quad \overline{\phi}_R(e_t, d_t, s_t) \leq \phi_t \\ (1-\chi) \Big(\phi_t - \underline{\phi}_R(e_t, d_t, s_t) \Big) (e_t + d_t)^\zeta & \text{if} \quad \underline{\phi}_R(e_t, d_t, s_t) \leq \phi_t \leq \overline{\phi}_R(e_t, d_t, s_t) \end{cases}$$

$$\underline{\phi}_R(e_t, d_t, s_t) = \begin{cases} 0 & \text{if} \quad 0 \leq (1+r_m) d_t (1-s_t) < (1-\chi)(1-\delta)(e_t + d_t) \\ I^{-1} \left(\frac{(1+r_m) d_t (1-s_t) - (1-\chi)(1-\delta)(e_t + d_t)}{(1-\chi)(e_t + d_t)^\zeta} ; e_t + d_t \right) & \text{if} \quad (1-\chi)(1-\delta)(e_t + d_t) \leq (1+r_m) d_t (1-s_t) \\ & \qquad \leq (1-\chi)(1-\delta)(e_t + d_t) + (1-\chi)(e_t + d_t)^\zeta \hat{I}(e_t + d_t) \end{cases}$$

$$\overline{\phi}_R(e_t, d_t, s_t) = \begin{cases} 0 & \text{if} \quad 0 \leq (1+r_b) d_t s_t < \chi(1-\delta)(e_t + d_t) \\ G^{-1} \left(\frac{(1+r_b) d_t s_t - \chi(1-\delta)(e_t + d_t)}{\chi (e_t + d_t)^\zeta} \right) & \text{if} \quad \chi(1-\delta)(e_t + d_t) \leq (1+r_b) d_t s_t \\ & \qquad \leq \chi(1-\delta)(e_t + d_t) + \chi(e_t + d_t)^\zeta \mathbb{E}(\phi) \end{cases}$$

and:

$$\forall e_t \in [0, E], \quad T_K V(e_t) \quad = \quad \max_{(d_t, s_t) \in \mathcal{S}_K(e_t)} \int_{\phi_t \geq \underline{\phi}_K(e_t, d_t, s_t)} V^c \left(n_K \left(\phi_t; e_t, d_t, s_t \right) \right) dF(\phi_t) \tag{A1-K}$$

$$\text{s.t.} \quad V^c(n_t) = \max_{0 \leq e_{t+1} \leq n_t} n_t - e_{t+1} + (1-\eta)\beta V(e_{t+1})$$

$$n_K(\phi_t, e_t, d_t, s_t) = \Big(\phi_t - \underline{\phi}_K(e_t, d_t, s_t) \Big) (e_t + d_t)^\zeta \qquad \text{if} \quad \phi_t \geq \underline{\phi}_K(e_t, d_t, s_t)$$

$$\underline{\phi}_K(e_t, d_t, s_t) = \begin{cases} 0 & \text{if} \quad 0 \leq (1 + r_m(1-s_t) + r_b s_t) d_t < (1-\delta)(e_t + d_t) \\ M^{-1} \left(\frac{(1 + r_m(1-s_t) + r_b s_t) d_t - (1-\delta)(e_t + d_t)}{(e_t + d_t)^\zeta} ; e_t + d_t \right) & \text{if} \quad (1-\delta)(e_t + d_t) \leq (1 + r_m(1-s_t) + r_b s_t) d_t \\ & \qquad \leq (1-\delta)(e_t + d_t) + (e_t + d_t)^\zeta \hat{M}(e_t + d_t) \end{cases}$$

Consider a solution V to problem **(M-8)** and a particular value of $e_t \in [0, E]$. Since $(\mathcal{S}_K(e_t), \mathcal{S}_R(e_t))$ is a partition of $\mathcal{S}(e_t)$, the optimal policies $(\hat{d}_t, \hat{s}_t)$ (there may be several) must be in either $S_R(e_t)$ or $S_K(e_t)$. Assume that they are in $\mathcal{S}_R(e_t)$. Then the contracts $R_{b,t}, R_{m,t}$ associated with the optimal policies satisfy $\frac{R_{b,t}}{\chi} \geq \frac{R_{m,t}}{1-\chi}$. Given the results of proposition 1, the constraints and objectives in problem **(M-8)** can be rewritten as in (A1-R). Since V solves **(M-8)**, this implies that $V(e_t) = T_R V(e_t)$. Moreover, in that case $T_R V(e_t) = V(e_t) \geq T_K V(e_t)$, by optimality of $(\hat{d}_t, \hat{s}_t)$. Thus, $TV(e_t) = V(e_t)$. The same equality obtains if $(\hat{d}_t, \hat{s}_t) \in \mathcal{S}_K(e_t)$. Any solution to V to problem **(M-8)** must thus satisfy $TV = V$. The rest of the proof therefore focuses on the properties of the operators T, T_K and T_R.

Step 2: Let $V \in C(E)$. By lemma 1, the associated continuation value V^c is continuous on $\mathbb{R}^+$. Moreover, since I^{-1} and G^{-1} are continuous functions of e_t, d_t and s_t, the functions n_R, $\underline{\phi}_R$ and $\overline{\phi}_R$ are continuous in their (e_t, d_t, s_t) arguments. Define the mapping $O_R : [0, E] \times \left[0, \bar{d}(E)\right] \times [0, 1] \to \mathbb{R}_+$ by $O_R(e_t, d_t, s_t) = \int_{\phi_t \geq \underline{\phi}_R(e_t, d_t, s_t)} V^c \left(n_t^R \left(\phi_t; e_t, d_t, s_t\right)\right) dF(\phi_t)$. Here $\bar{d}(E)$ denotes the upper bound on borrowing for the maximum level of equity E; see Crouzet (2013), proposition 10, for a proof that such an upper bound always exist. By continuity of $V^c, n_t^R, \underline{\phi}_R$ and $\overline{\phi}_R$, the integrand in O_R is continuous on the compact set $[0, E] \times \left[0, \bar{d}(E)\right] \times [0, 1]$, and therefore uniformly continuous. Hence, O_R is continuous on $[0, E] \times \left[0, \bar{d}(E)\right] \times [0, 1]$. The constraint correspondence $\Gamma_R : e_t \to \mathcal{S}_R(e_t)$ maps $[0, E]$ into $\left[0, \bar{d}(E)\right] \times [0, 1]$. The characterization of the set $S_R(e_t)$ in Crouzet (2013), proposition 2, moreover shows that the graph of the correspondence Γ_R is closed and convex. Theorems 3.4 and 3.5 in Stokey, Lucas, and Prescott (1989) then indicate that Γ_R is continuous. Given that O_R is continuous and Γ_R compact-valued and continuous, the theorem of the maximum applies, and guarantees that $T_R V \in C(E)$. In analogous steps, one can prove that $T_K V \in C(E)$. Therefore, $TV = \max(T_R V, T_K V) \in C(E)$. Moreover, let $V \in C_0(E)$. Then $V^c(0) \geq 0$ and V^c is increasing, by lemma 1. Moreover $S_R(0) \neq \emptyset$, so one can evaluate O_R at some $(d_{t,0}, s_{t,0}) \in S_R(0)$. Since $n_R \geq 0$, $V^c(0) \geq 0$ and V^c is increasing, $O_R(0, d_{t,0}, s_{t,0}) \geq 0$. Therefore $T_R V(0) \geq 0$, so $TV(0) \geq 0$ and $TV \in C_0(E)$.

Step 3: Finally, I establish that the operator T has the monotonicity and discounting properties. First, let $(V, W) \in C(E)$ such that $\forall e_t \in C(E)$, $V(e_t) \geq W(e_t)$. Pick a particular $e_t \in [0, E]$. By an argument similar to the proof of lemma 1, $\forall n_t \geq 0, V^c(n_t) \geq W^c(n_t)$, where W^c denotes the solution to the dividend issuance problem when the continuation value is W (and analogously for V). Since the functions $\underline{\phi}_R, \overline{\phi}_R$ and n_R are independent of V, this inequality implies $O_R^V(e_t, d_t, s_t) \geq O_R^W(e_t d_t, s_t)$ for any $(d_t, s_t) \in \mathcal{S}_R(e_t)$, where the notation O_R^W designates the objective function in problem (A1-R) when the continuation value function is W (and analogously for V). Thus $T_R V(e_t) \geq T_R W(e_t)$. Similarly, one can show that $T_K V(e_t) \geq T_K W(e_t)$.

Therefore, $TV(e_t) \geq TW(e_t)$, and T has the monotonicity property. To establish the discounting property, it is sufficient to note that $(V+a)^c(n_t) = V^c(n_t) + \beta a$, so that for any $e_t \in [0, E]$ and $(d_t, s_t) \in \mathcal{S}_R(e_t)$, $O_R^{V+a}(e_t, d_t, s_t) = O_R^V(e_t, d_t, s_t) + \left(1 - F\left(\underline{\phi}_R(e_t, d_t, s_t)\right)\right)\beta a \leq O_R^V(e_t, d_t, s_t) + \beta a$. This shows that $T_R(V + a)(e_t) \leq T_R V(e_t) + \beta a$. A similar claim can be made for T_K. Therefore, the operator T has the discounting property. The Blackwell sufficiency conditions hold, so that T is a contraction mapping. ■

Properties of the solution to problem (M-8).

Let V denote the unique solution to problem **(M-8)**.

Monotonicity First, it can be shown that $\forall (e_t^1, e_t^2) \in [0, E]$ s.t. $e_t^1 < e_t^2$, $T_R V(e_t^1) < T_R V(e_t^2)$. To show this, let $(d_t, s_t) \in S_R(e_t^1)$. I proceed in three steps:

1 : First, since $e_t^1 < e_t^2$, $S_R(e_t^1) \subset S_R(e_t^2)$ (a proof for this can be obtained using proposition 10 in Crouzet (2013); intuitively, this result indicates that increasing net worth relaxes borrowing constraints). Therefore, $(d_t, s_t) \in S_R(e_t^2)$.

2 : Next, I show that $\overline{\phi}_R(e_t^1, d_t, s_t) > \overline{\phi}_R(e_t^2, d_t, s_t)$. (The functions at this point are well-defined because $(d_t, s_t) \in S_R(e_t^2)$). Given the expression for $\overline{\phi}_R$ in (A1-R), since G^{-1} is strictly increasing on $\mathbb{R}_+$, it is sufficient to show that $e_t \to \frac{(1+r_b)d_t s_t - \chi(1-\delta)(e_t+d_t)}{\chi(e_t+d_t)^\zeta}$ is strictly decreasing in e_t. This is true because $\zeta < 1$. Next I prove that $\underline{\phi}_R(e_t^1, d_t, s_t) > \underline{\phi}_R(e_t^2, d_t, s_t)$. To see this, note that:

$$\frac{\partial \underline{\phi}_R(e_t, d_t, s_t)}{\partial e_t} = \frac{\frac{\partial y_I}{\partial e_t} - I_2\left(I^{-1}\left(y_I(e_t); e_t + d_t\right); e_t + d_t\right)}{I_1\left(I^{-1}\left(y_I(e_t); e_t + d_t\right); e_t + d_t\right)},$$

where $y_I(e_t) \equiv \frac{(1+r_m)d_t(1-s_t) - (1-\chi)(1-\delta)(e_t+d_t)}{(1-\chi)(e_t+d_t)^\zeta}$. Note that, letting $x = I^{-1}\left(y_I(e_t); e_t + d_t\right)$:

$$\frac{\partial y_I}{\partial e_t} - I_2(x; e_t + d_t) = -\zeta \frac{(1+r_m)d_t(1-s_t)}{(1-\chi)(e_t+d_t)^{\zeta+1}} - (1-\zeta)(1-\delta)(e_t+d_t)^{-\zeta}\left(1 - F(x)\right) < 0$$

Since $I_1 > 0$, this implies that $\underline{\phi}_R$ is strictly decreasing in e_t.

3 : Given the fact that $\underline{\phi}_R$ and $\overline{\phi}_R$ are strictly decreasing in e_t, the expression for $n_R(\phi_t; e_t, d_t, s_t)$ in (A1-R) then implies that $\forall \phi_t \geq \underline{\phi}_R(e_t^1, d_t, s_t)$, $n_R(\phi_t; e_t^1, d_t, s_t) < n_R(\phi_t; e_t^2, d_t, s_t)$. Thus, since V^c is increasing,

$$
\begin{aligned}
O_R(e_t^1, d_t, s_t) = \int_{\phi_t \geq \underline{\phi}_R(e_t^1, d_t, s_t)} V^c\left(n_R\left(\phi_t; e_t^1, d_t, s_t\right)\right) dF(\phi_t) \quad &< \quad \int_{\phi_t \geq \underline{\phi}_R(e_t^1, d_t, s_t)} V^c\left(n_R\left(\phi_t; e_t^2, d_t, s_t\right)\right) dF(\phi_t) \\
&\leq \quad \int_{\phi_t \geq \underline{\phi}_R(e_t^2, d_t, s_t)} V^c\left(n_R\left(\phi_t; e_t^2, d_t, s_t\right)\right) dF(\phi_t) \\
&= \quad O_R(e_t^2, d_t, s_t).
\end{aligned}
$$

where the second line exploits the fact that $\underline{\phi}_R(e_t^1, d_t, s_t) > \underline{\phi}_R(e_t^2, d_t, s_t)$ and $V^c \geq 0$.

Since last inequality has been established for any $(d_t, s_t) \in S_R(e_t^1) \subset S_R(e_t^2)$, it shows that the objective function is uniformly increasing (strictly) in e_t, so that $T_R V(e_t^1) < T_R V(e_t^2)$. A similar but simpler proof using the expression for $\underline{\phi}_K$ in (A1-K) shows that $T_K V(e_t^1) < T_K V(e_t^2)$. Therefore, $TV(e_t^1) < TV(e_t^2)$, so that $V(e_t^1) < V(e_t^2)$. Therefore, the solution to problem **(M-8)** is strictly increasing in e_t. ■

Existence and unicity of an invariant measure of firms across equity levels.

I next prove that, given a solution to problem **(M-8)**, an measure of firms across levels of e_t exists and is unique. I start by introducing some preliminary notation.

Preliminary notation $\bar{e}$ denotes the level of net worth above which firms start issuing dividends. $\bar{E} = [0, \bar{e}]$ denotes the state-space of the firm problem **(M-8)**. $\left(\bar{E}, \bar{\mathcal{E}}\right)$ is the measurable space composed of $\bar{E}$ and the family of Borel subsets of $\bar{E}$. For any value $e_t \in \bar{E}$, $\hat{d}(e_t)$ and $\hat{s}(e_t)$ denote the policy functions of the firms. The fact that these policy functions are such that $(\hat{d}(e_t), \hat{s}(e_t)) \in \mathcal{S}_R(e_t)$ will be denoted by $e_t \in \bar{E}_R$, and $e_t \in \bar{E}_K$ for the other case. $\hat{\underline{\phi}}_R(e_t)$ and $\hat{\bar{\phi}}_R(e_t)$ denote the liquidation threshold implied by the firm's policy functions when $e_t \in \bar{E}_R$, while $\hat{\underline{\phi}}_K(e_t)$ denotes the liquidation threshold when $e_t \in \bar{E}_K$. I use the notation: $\hat{r}(e_t) = r_m(1 - \hat{s}(e_t)) + r_b \hat{s}(e_t)$. Finally, recall that $F(.)$ denotes the CDF of ϕ_t, the idiosyncratic productivity shock, and η denotes the exogenous exit probability.

Transition function To define the transition function Q implied by firms' policy functions, one can proceed by constructing the probability of a firm having an equity level smaller than or equal to e_{t+1} next period, given that its current period equity is e_t. Additionally, one must take into account the fact that the fraction η of firms that exit exogenously, plus those that are liquidated endogenously, will be replaced by firms operating at the entry scale e^e. Insofar as the evolution of the measure of firms is concerned, this is equivalent to assuming that firms these firms transition to the level e^e.[7] The resulting probability of having

[7]However, one cannot assume this in the expression of the value function of the firm; this would indeed change firms' incentive to default, restructure and renegotiate.

an equity level smaller than or equal to e_{t+1}, given that current period equity is e_t is denoted by $N(e_t, e_{t+1})$, and is given by the following expressions.

$\boxed{\bullet\ \textbf{For } e_t \in \bar{E}_K\text{:}}$

- If $(1-\delta)(e_t + \hat{d}(e_t)) > (1+\hat{r}(e_t))\hat{d}(e_t) + \bar{e}$, the firm never liquidates and always has a net worth of at least $\bar{e}$ after the debt settlement stage, so:

$$N(e_t, e_{t+1}) = \eta \mathbb{1}\{e^e \le e_{t+1}\} + (1-\eta) \begin{cases} 0 & \text{if } e_{t+1} \le \bar{e} \\ 1 & \text{if } \bar{e} \le e_{t+1} \end{cases}$$

- If $(1+\hat{r}(e_t))\hat{d}(e_t) + \bar{e} \ge (1-\delta)(e_t + \hat{d}(e_t)) > (1+\hat{r}(e_t))\hat{d}(e_t)$, the firm is never liquidated but may have a net worth below $\bar{e}$ after the debt settlement stage, so:

$$N(e_t, e_{t+1}) = \eta \mathbb{1}\{e^e \le e_{t+1}\} + (1-\eta) \begin{cases} 0 & \text{if } e_{t+1} < \underline{e}(e_t) \\ F\left(\frac{e_{t+1} - \underline{e}(e_t)}{(e_t + \hat{d}(e_t))^\varsigma}\right) & \text{if } \underline{e}(e_t) \le e_{t+1} < \bar{e} \\ 1 & \text{if } \bar{e} \le e_{t+1} \end{cases}$$

where $\underline{e}(e_t) = (1-\delta)(e_t + \hat{d}(e_t)) - (1+\tilde{r}(e_t))\hat{d}(e_t)$ is the lower bound on these firms' net worth (that is, their net net worth when $\phi_t = 0$).

- If $(1+\hat{r}(e_t))\hat{d}(e_t) \ge (1-\delta)(e_t + \hat{d}(e_t))$, the firm may be liquidated at the debt settlement stage (when $\phi_t \le \hat{\underline{\phi}}_K(e_t)$), so:

$$\begin{aligned} N(e_t, e_{t+1}) &= \left(\eta + (1-\eta)F\left(\hat{\underline{\phi}}_K(e_t)\right)\right)\mathbb{1}\{e^e \le e_{t+1}\} \\ &\quad + (1-\eta) \begin{cases} 0 & \text{if } e_{t+1} < 0 \\ F\left(\hat{\underline{\phi}}_K(e_t) + \frac{e_{t+1}}{(e_t + \hat{d}(e_t))^\varsigma}\right) - F\left(\hat{\underline{\phi}}_K(e_t)\right) & \text{if } 0 \le e_{t+1} < \bar{e} \\ 1 - F\left(\hat{\underline{\phi}}_K(e_t)\right) & \text{if } \bar{e} \le e_{t+1} \end{cases} \end{aligned}$$

$\boxed{\bullet\ \textbf{For } e_t \in \bar{E}_R\text{:}}$ There are two subcases, depending on whether $\hat{\underline{\phi}}_R(e_t) \gtrless \hat{\hat{\phi}}_R(e_t) - \frac{\bar{e}}{(1-\chi)(e_t + \hat{d}(e_t))^\varsigma}$.

$\boxed{\textbf{- If } \hat{\underline{\phi}}_R(e_t) > \hat{\hat{\phi}}_R(e_t) - \frac{\bar{e}}{(1-\chi)(e_t + \hat{d}(e_t))^\varsigma}\textbf{:}}$

- If $(1-\delta)(e_t + \hat{d}(e_t)) \ge (1+\hat{r}(e_t))\hat{d}(e_t) + \bar{e}$, the firm never liquidates or restructures, and always has a net worth of at least $\bar{e}$ after the debt settlement stage, so:

$$N(e_t, e_{t+1}) = \eta \mathbb{1}\{e^e \le e_{t+1}\} + (1-\eta) \begin{cases} 0 & \text{if } e_{t+1} \le \bar{e} \\ 1 & \text{if } \bar{e} \le e_{t+1} \end{cases}$$

- If $(1+\hat{r}(e_t))\hat{d}(e_t)+\bar{e} > (1-\delta)(e_t+\hat{d}(e_t)) \geq \frac{(1+r_b)\hat{s}(e_t)\hat{d}(e_t)}{\chi}$, the firm may sometimes have a net worth below $\bar{e}$ after the debt settlement stage, but still never liquidates or restructures, so:

$$N(e_t, e_{t+1}) = \eta\mathbb{1}\{e^e \leq e_{t+1}\} + (1-\eta)\begin{cases} 0 & \text{if } e_{t+1} < \underline{e}(e_t) \\ F\left(\frac{e_{t+1}-\underline{e}(e_t)}{(e_t+\hat{d}(e_t))^\zeta}\right) & \text{if } \underline{e}(e_t) \leq e_{t+1} < \bar{e} \\ 1 & \text{if } \bar{e} \leq e_{t+1} \end{cases}$$

where again, $\underline{e}(e_t) = (1-\delta)(e_t+\hat{d}(e_t)) - (1+\hat{r}(e_t))\hat{d}(e_t)$.

- If $\frac{(1+r_b)\hat{s}(e_t)\hat{d}(e_t)}{\chi} > (1-\delta)(e_t+\hat{d}(e_t)) \geq \frac{(1+r_m)(1-\hat{s}(e_t))}{1-\chi}$, then the firm will sometimes restructure at the debt settlement stage, but never liquidate, so:

$$N(e_t, e_{t+1}) = \eta\mathbb{1}\{e^e \leq e_{t+1}\} + (1-\eta)\begin{cases} 0 & \text{if } e_{t+1} < \underline{e}(e_t) \\ F\left(\frac{e_{t+1}-\underline{e}(e_t)}{(1-\chi)(e_t+\hat{d}(e_t))^\zeta}\right) & \text{if } \underline{e}(e_t) \leq e_{t+1} < \underline{e}(e_t) + (1-\chi)(e_t+\hat{d}(e_t))^\zeta \hat{\bar{\phi}}_R(e_t) \\ F\left(\frac{e_{t+1}-\underline{e}(e_t)}{(e_t+\hat{d}(e_t))^\zeta} + \chi\hat{\bar{\phi}}_R(e_t)\right) & \text{if } \underline{e}(e_t) + (1-\chi)(e_t+\hat{d}(e_t))^\zeta \hat{\bar{\phi}}_R \leq e_{t+1} < \bar{e} \\ 1 & \text{if } \bar{e} \leq e_{t+1} \end{cases}$$

where now, $\underline{e}(e_t) = (1-\chi)(1-\delta)(e_t+\hat{d}(e_t)) - (1+r_m)(1-\hat{s}(e_t))\hat{d}(e_t)$.

- Finally, if $\frac{(1+r_m)(1-\hat{s}(e_t))\hat{d}(e_t)}{(1-\chi)} > (1-\delta)(e_t+\hat{d}(e_t))$, then the firm will sometimes be liquidated, sometimes restructure and sometimes repay, so:

$$\begin{aligned} N(e_t, e_{t+1}) \quad = \quad & \left(\eta + (1-\eta)F\left(\hat{\underline{\phi}}_R(e_t)\right)\right)\mathbb{1}\{e^e \leq e_{t+1}\} \\ & +(1-\eta)\begin{cases} 0 & \text{if } e_{t+1} < 0 \\ F\left(\hat{\underline{\phi}}_R(e_t) + \frac{e_{t+1}}{(1-\chi)(e_t+\hat{d}(e_t))^\zeta}\right) - F\left(\hat{\underline{\phi}}_R(e_t)\right) & \text{if } 0 \leq e_{t+1} < (1-\chi)(e_t+\hat{d}(e_t))^\zeta\left(\hat{\bar{\phi}}_R(e_t) - \hat{\underline{\phi}}_R(e_t)\right) \\ F\left(\frac{e_{t+1}}{(e_t+\hat{d}(e_t))^\zeta} + \chi\hat{\bar{\phi}}_R(e_t) + (1-\chi)\hat{\underline{\phi}}_R(e_t)\right) - F\left(\hat{\underline{\phi}}_R(e_t)\right) & \text{if } (1-\chi)(e_t+\hat{d}(e_t))^\zeta\left(\hat{\bar{\phi}}_R(e_t) - \hat{\underline{\phi}}_R(e_t)\right) \leq e_{t+1} < \bar{e} \\ 1 - F\left(\hat{\underline{\phi}}_R(e_t)\right) & \text{if } \bar{e} \leq e_{t+1} \end{cases} \end{aligned}$$

- If $\hat{\underline{\phi}}_R(e_t) \leq \hat{\bar{\phi}}_R(e_t) - \frac{\bar{e}}{(1-\chi)(e_t+\hat{d}(e_t))^\zeta}$:

- If $(1-\delta)(e_t+\hat{d}(e_t)) \geq \frac{(1+r_m)(1-\hat{s}(e_t))\hat{d}(e_t)+\bar{e}}{1-\chi}$, then even when the firm restructures, it will still have enough net worth to have an equity level $\bar{e}$ tomorrow, so that:

$$N(e_t, e_{t+1}) = \eta\mathbb{1}\{e^e \leq e_{t+1}\} + (1-\eta)\begin{cases} 0 & \text{if } e_{t+1} \leq \bar{e} \\ 1 & \text{if } \bar{e} \leq e_{t+1} \end{cases}$$

- If $\frac{(1+r_m)(1-\hat{s}(e_t))\hat{d}(e_t)+\bar{e}}{1-\chi} > (1-\delta)(e_t+\hat{d}(e_t)) \geq \frac{(1+r_m)(1-\hat{s}(e_t))\hat{d}(e_t)}{1-\chi}$, then the firm may restructure and not have sufficient equity to reach the level $\bar{e}$ tomorrow, so that:

$$N(e_t, e_{t+1}) = \eta \mathbb{1}\{e^e \leq e_{t+1}\} + (1-\eta) \begin{cases} 0 & \text{if } e_{t+1} < \underline{e}(e_t) \\ F\left(\frac{e_{t+1} - \underline{e}(e_t)}{(1-\chi)(e_t + \hat{d}(e_t))^{\varsigma}}\right) & \text{if } \underline{e}(e_t) \leq e_{t+1} < \bar{e} \\ 1 & \text{if } \bar{e} \leq e_{t+1} \end{cases}$$

where again $\underline{e}(e_t) = (1-\chi)(1-\delta)(e_t + \hat{d}(e_t)) - (1+r_m)(1-\hat{s}(e_t))\hat{d}(e_t)$ is the lower bound on these firms' net worth (that is, their net net worth when $\phi_t = 0$).

- Finally, if $\frac{(1+r_m)(1-\hat{s}(e_t))\hat{d}(e_t)}{1-\chi} \geq (1-\delta)(e_t + \hat{d}(e_t))$, then the firm will sometimes liquidate, so that:

$$\begin{aligned} N(e_t, e_{t+1}) &= \left(\eta + (1-\eta) F\left(\underline{\hat{\phi}}_R(e_t)\right)\right) \mathbb{1}\{e^e \leq e_{t+1}\} \\ &\quad +(1-\eta) \begin{cases} 0 & \text{if } e_{t+1} < 0 \\ F\left(\underline{\hat{\phi}}_R(e_t) + \frac{e_{t+1}}{(1-\chi)(e_t + \hat{d}(e_t))^{\varsigma}}\right) - F\left(\underline{\hat{\phi}}_R(e_t)\right) & \text{if } 0 \leq e_{t+1} < \bar{e} \\ 1 - F\left(\underline{\hat{\phi}}_R(e_t)\right) & \text{if } \bar{e} \leq e_{t+1} \end{cases} \end{aligned}$$

It is straightforward to check that given e_t, $N(e_t, .)$ is weakly increasing, has limits 0 and 1 at $-\infty$ and $+\infty$ and is everywhere continuous from above, using the expressions given above. Following theorem 12.7 of Stokey, Lucas, and Prescott (1989), there is therefore a unique probability measure $\hat{Q}(e_t, .)$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\hat{Q}(e_t,]-\infty, e]) = N(e_t, e)\ \forall e \in \mathbb{R}$. This measure is 0 for $e < 0$ and 1 for $e > \bar{e}$, and so the restriction of $\hat{Q}$ to $\bar{\mathcal{E}}$ is also a probability measure on $\left(\bar{E}, \bar{\mathcal{E}}\right)$, which can be denoted $Q(e_t, .)$. Moreover, fixing $e \in \bar{E}$, the function $Q(.,]-\infty, e]) : \bar{E} \rightarrow [0,1]$ is measurable with respect to $\bar{\mathcal{E}}$. Indeed, given $z \in [0,1]$, the definition of $N(e_t, e)$ indicates that the set $H(z, e) = \left\{e_t \in \bar{E} \text{ s.t. } Q(e_t, [0, e]) \leq z\right\} = \left\{e_t \in \bar{E} \text{ s.t. } \hat{Q}(e_t,]-\infty, e]) \leq z\right\} = \left\{e_t \in \bar{E} \text{ s.t. } N(e_t, e) \leq z\right\}$ is an element of $\bar{\mathcal{E}}$.[8] This in turn implies that the function $Q(., \mathbf{E})$ is $\bar{\mathcal{E}}$-measurable for any $\mathbf{E} \in \bar{\mathcal{E}}$. The function $Q : \bar{E} \times \bar{\mathcal{E}} \rightarrow [0,1]$ is such that $Q(e_t, .)$ is therefore a probability measure for any $e_t \in \bar{E}$, and $Q(., \mathbf{E})$ is $\bar{\mathcal{E}}$-measurable $\forall \mathbf{E} \in \bar{\mathcal{E}}$; hence, Q is a transition function.

[8] For example, if $z \leq \eta$ and $0 < e < \bar{e}$, the intersection of the set $H(z, e)$ with $\bar{E}_K$ is given by:

$$\begin{aligned} H(z,e) \cap \bar{E}_K &= \left\{e_t \in \bar{E}_K \text{ s.t. } (1-\delta)(e_t + \hat{d}(e_t)) \geq (1+\hat{r}(e_t))\hat{d}(e_t) + \bar{e}\right\} \cup \\ &\qquad \left\{e_t \text{ s.t. } (1+\hat{r}(e_t))\hat{d}(e_t) + \bar{e} > (1-\delta)(e_t + \hat{d}(e_t)) \geq (1+\hat{r}(e_t))\hat{d}(e_t) \text{ and } \underline{e}(e_t) > e\right\} \\ &= \left\{e_t \in \bar{E}_K \text{ s.t. } (1-\delta)(e_t + \hat{d}(e_t)) > (1+\tilde{r}(e_t))\hat{d}(e_t) + e\right\} \end{aligned}$$

This set is the inverse image of $]0, +\infty[$ by the function $g : \bar{E}_K \rightarrow \mathbb{R}, x \rightarrow (1-\delta)(x + \hat{d}(x)) - (1+\tilde{r}(x))\hat{d}(x) - e$. Since the policy functions are continuous, g is continuous. Since the inverse image of an open set by a continuous function is an open set, $H(z,e) \cap \bar{E}_K$ is an open set, and hence a Borel set. The intersection $H(x,e) \cap \bar{E}_R$ has a more complicated expression, but also boils down to a finite union of sets that are open because of the continuity of policy functions. Given that $\bar{E}_K$ and $\bar{E}_R$ are intervals that form a partition of $\bar{E}$, this implies that $H(z,e) = (H(z,e) \cap \bar{E}_R) \cup (H(z,e) \cup \bar{E}_K)$ is a finite union of Borel sets, and therefore a Borel set. This line of reasoning applies for all $0 \leq z \leq 1$ and $0 \leq e \leq \bar{e}$.

Feller property To establish that the transition function Q has the Feller property, one must show that $\forall e_t \in \bar{E}$ and $\forall$ $(e_{n,t})_n \in \bar{E}^{\mathbb{N}}$ such that $e_{n,t} \to e_t$, $Q(e_{n,t}, .) \Rightarrow Q(e_t, .)$, where $\Rightarrow$ denotes weak convergence. To establish this, given the definition of Q it is sufficient to show that $N(e_{n,t}, e_{t+1}) \to N(e_t, e_{t+1})$ pointwise, at all values of e_{t+1} and e_t where $N(., e_{t+1})$ and $N(e_t, .)$ are continuous. This excludes, in particular, the cases $e_{t+1} = \bar{e}$ or $e_t = \bar{e}$. I outline the proof for the case $e_t \in \bar{E}_K$; the proof for the other case is similar.

First consider a simple case, when e_t satisfies:

$$(1 + \hat{r}(e_t))\hat{d}(e_t) > (1 - \delta)(e_t + \hat{d}(e_t)) \tag{1}$$

Because $e_t \mapsto \eta + (1-\eta)F\left(\hat{\underline{\phi}}_K(e_t) + \frac{e_{t+1}}{(e_t + \hat{d}(e_t))^{\zeta}}\right)$ is continuous on $\bar{E}_K$, $e \mapsto N(e, e_{t+1})$ is continuous in a neighborhood of e_t. Additionally, since $\hat{r}$ and $\hat{d}$ is continuous, the inequality (1) holds for $e_{n,t}$, when n is sufficiently large. Combining these two observations implies that $N(e_{n,t}, e_{t+1}) \to N(e_t, e_{t+1})$. The case when the inequality above holds in the reverse direction is handled similarly.

Now consider a knife-edge case:

$$(1 + \hat{r}(e_t))\hat{d}(e_t) = (1 - \delta)(e_t + \hat{d}(e_t)) \tag{2}$$

The proglem is that the sequence $(e_{n,t})_n$ can have elements that satisfy either $(1 + \hat{r}(e_{n,t}))\hat{d}(e_{n,t} \gtrless (1 - \delta)(e_{n,t} + \hat{d}(e_{n,t})$, which correspond to different expressions for $N(e_{n,t}, e_{t+1})$; one must check therefore check that $N(., e_{t+1})$ is continuous at e_t such that (2) holds. At such a point, $\underline{e}(e_t) = 0$, so that:

$$\lim_{e \uparrow e_t} N(e_t, e_{t+1}) \quad = \quad \lim_{e \uparrow e_t} \eta + (1-\eta)F\left(\frac{e_{t+1} - \bar{e}(e_t)}{(e_t + \hat{d}(e_t))^{\zeta}}\right) = \eta + (1-\eta)F\left(\frac{e_{t+1}}{(e_t + \hat{d}(e_t))^{\zeta}}\right).$$

Moreover, $\hat{\underline{\phi}}_K(e_t) = 0$, so that:

$$\lim_{e \downarrow e_t} N(e_t, e_{t+1}) \quad = \quad \lim_{e \downarrow e_t} \eta + (1-\eta)F\left(\hat{\underline{\phi}}_K(e_t) + \frac{e_{t+1}}{(e_t + \hat{d}(e_t))^{\zeta}}\right) = \eta + (1-\eta)F\left(\frac{e_{t+1}}{(e_t + \hat{d}(e_t))^{\zeta}}\right).$$

This establishes the continuity of $N(., e_{t+1})$ at points e_t such that equation (2) holds; thus, $\forall e_{t+1} \in \bar{E}$, $\bar{e}_{t+1} \neq \bar{e}$, $N(., e_{t+1})$ is continuous on $\bar{E}_K$. ■

3 Computational procedures

Computation of a solution to the firm's problem (M-8) The algorithm is a straightforward iteration in two fixed points, $V(.)$ and $\bar{e}$, except for the important insight that the feasible sets $\mathcal{S}_K(e_t)$ and $\mathcal{S}_R(e_t)$

can be computed *outside* of iterations used to compute fixed point of the mapping T (value functions $V(.)$). This is again because the feasible sets are independent of the value function $V(.)$, but only depend on firms' equity e_t.[9]

1. Guess a value for $\bar{e}$ and choose a discrete grid with N_e points on $[0, \bar{e}]$, $\{e_{i,t}\}_{i=1}^{N_e}$.[10]

2. For $i = 1, ..., N_e$, compute and store the frontiers of the sets $\mathcal{S}_K(e_{i,t})$ and $\mathcal{S}_R(e_{i,t})$ depicted in figure **(M-4)**. This computation is done using the results in Crouzet (2013). For example, the frontiers of the set $\mathcal{S}_R(e_{i,t})$ are given by $\left\{\left(s, \bar{d}_R(s)\right) \in [0,1] \times \mathbb{R}^+ \text{ s.t. } s \in [s_{R,min}, 1]\right\}$, where $s_{R,min} = \frac{1}{1+\frac{1-\chi}{\chi}\frac{1+r_b}{1+r_m}}$ and $\bar{d}_R(.)$ is a function of s, the expression of which is reported in Crouzet (2013). The frontiers are computed by discretizing the interval $[s_{R,min}, 1]$ on a grid of size N_s, evaluating the function $\bar{d}_R(.)$ at each of these points, and storing the resulting points $\{s_j, \bar{d}(s_j)\}_{j=1}^{N_s}$.[11] The frontier of $\mathcal{S}_K(e_{i,t})$ is similarly approximated on a discrete grid covering $[0, s_{R,max}(e_{i,t})]$.

3. Choose an initial guess for the value function $V_0(.)$ at the points $\{e_{i,t}\}$. A useful initial guess is to first compute the value $V_S(.)$ of the firm in the static version of the model, and then use $V_0(e_{i,t}) = V_S(e_{i,t}) + \frac{1}{1-(1-\eta)\beta}\left(\mathbb{E}(\phi)(\bar{e} + \hat{d}_S(\bar{e}))^\zeta + (1-\delta)(\bar{e} + \hat{d}_S(\bar{e})) - (1+r_m)\hat{d}_S(\bar{e}) - \bar{e}\right)$, where $\hat{d}_S(\bar{e})$ denotes total borrowing when $e = \bar{e}$ in the solution to the static model. The term in parenthesis represents dividend issuance by an infinitely-lived firm with equity $\bar{e}$, market borrowing $\hat{d}_S(\bar{e})$ and no bank loans, assuming that this borrowing amount is riskless.

4. Apply the mapping T defined in equation (A1) until convergence. This step takes into account the fact that the firm chooses market-only borrowing for some $e_t \geq e^*$. It proceeds as follows:

   (a) Guess a value for e^*, the switching threshold, and for $\left\{\hat{d}(e_{i,t}), \hat{s}(e_{i,t})\right\}$, the firm's policy functions, and an initial guess for the value function $V^{(0)}$.

   (b) Given a guess for $V^{(n)}(.)$, compute the implied values $TV_R^{(n)}(e_{i,t})$ for $e_{i,t} \leq e^*$ and $TV_K^{(n)}(e_{i,t})$ for $e_{i,t} \geq e^*$, assuming that the policy of the firm is given by $\left\{\hat{d}(e_{i,t}), \hat{s}(e_{i,t})\right\}$. Let $V^{(n+1)}(e_{i,t}) = TV_R^{(n)}(e_{i,t})$ for $e_{i,t} \geq e^*$ and $V^{(n+1)}(e_{i,t}) = TV_K^{(n)}(e_{i,t})$ for $e_{i,t} > e^*$. Repeat this Howard improvement step for $0 \leq n \leq N_H$.

   (c) Given $V^{(N_H)}(.)$, compute policy functions, value functions, and the implied value of the switching threshold $e^{*,(N_H)}$.

[9] The computation of the solution to the problem **(M-8)** would otherwise be considerably more expensive.
[10] The grid used is log-spaced and concentrates points close to 0.
[11] N_s is taken to be sufficiently large to ensure that space between two gridpoints is at most 0.0005, which corresponds to 0.05% in terms of the share of bank credit as a fraction of total loans.

Steps (a)-(c) are started with the guess $V^{(0)} = V_0$. They are repeated until $V^{(N_H)}$ and $V^{(0)}$ are sufficiently close, and $e^{*,(0)}$ and $e^{*,(N_H)}$ are sufficiently close. At each repetition of $(a)-(c)$ after the initial one, the initial guesses used in (a) are the policy functions along with $TV^{(N_H)}$ and $e^{*,(N_H)}$ from the previous step. In steps (a)-(c), one needs to compute integrals of the form $\int_\phi V^c(n(\phi,d,s,e))dF(\phi)$, given the discrete approximation to the value function (which take into account the dividend issuance policy of the firm). I detail the method used below. The maximization over (d,s) in step (c) is carried out using a numerical constrained maximization procedure, rather than a discretized grid; the constraints are the frontiers computed in step 1.

5. Given the final value function $V^{(N_H)}(.)$, approximate numerically the derivative $\frac{\partial V^{(N_H)}}{\partial e}(\bar{e})$ and check whether it is sufficiently close to $\frac{1}{(1-\eta)\beta}$. If not, adjust the guess for $\bar{e}$ (upwards if $(1-\eta)\beta\frac{\partial V^{(N_H)}}{\partial e}(\bar{e}) > 1$, downwards otherwise) and repeat steps 1-4. To accelerate convergence, interpolate the value function $V^{(N_H)}(.)$ on the new grid $[0,\bar{e}]$ and use the interpolated value as new starting guess in step 3.

Computation of the value function A problem in steps (b) and (c) is the computation of the iterate TV, given a guess for V and values for the debt structure (d_t, s_t). Here, I give an example of the computation of this value function when $(d_t,s_t) \in \mathcal{S}_R(e_t)$ and: $\frac{(1+r_m)(1-s_t)d_t}{1-\chi} > (1-\delta)(e_t+d_t)$. This is the most involved case, since in that region, the firm is sometimes liquidated, sometimes manages to restructure its bank loans, and sometimes repays its creditors in full.

Taking into account the dividend issuance policy of the firm, as well as the liquidation/restructuring choice, the objective function of the firm (A1-R) is given by:

$$\int_{\phi_t \geq \underline{\phi}_R(e_t,d_t,s_t)} V^c\left(n_R\left(\phi_t;e_t,d_t,s_t\right)\right)dF(\phi_t) = (1-\eta)\beta\int_{\underline{\phi}_R(e_t,d_t,s_t)}^{\overline{\phi}_R(e_t,d_t,s_t)} V\left((\phi-\underline{\phi}_R(e_t,d_t,s_t))(1-\chi)(e_t+d_t)^\zeta\right)dF(\phi) \tag{INT-1}$$

$$+(1-\eta)\beta\int_{\overline{\phi}_R(e_t,d_t,s_t)}^{\phi^M(e_t,d_t,s_t,\bar{e})} V\left((\phi-\chi\overline{\phi}_R(e_t,d_t,s_t))-(1-\chi)\underline{\phi}_R(e_t,d_t,s_t))(e_t+d_t)^\zeta\right)dF(\phi) \tag{INT-2}$$

$$+\left(1-F(\phi^M(e_t,d_t,s_t,\bar{e}))\right)\left((1-\eta)\beta V(\bar{e})-\phi^M(e_t,d_t,s_t,\bar{e})(e_t+d_t)^\zeta\right)$$
$$+\left(\int_{\phi^M(e_t,d_t,s_t,\bar{e})}^{+\infty}\phi dF(\phi)\right)(e_t+d_t)^\zeta$$

where $\phi^M(e_t,d_t,s_t,\bar{e}) = \chi\overline{\phi}_R(e_t,d_t,s_t)) + (1-\chi)\underline{\phi}_R(e_t,d_t,s_t) + \frac{\bar{e}}{(e_t+d_t)^\zeta}$ is the threshold for the idiosyncratic shock ϕ above which the firm starts issuing dividends. The intervals $(INT-1)$ and $(INT-2)$ are computed using the linear interpolation of $V(.)$ on two different sub-intervals. Namely, first find the index $1 \leq k_0 \leq n-1$

such that $(1-\chi)(e_t+d_t)^\zeta(\overline{\phi}_R(e_t,d_t,s_t)-\underline{\phi}_R(e_t,d_t,s_t)) \in]e_{t,k_0};e_{t,k_0+1}]$. Define $\{\phi_k\}_{k=1}^{N_e}$, $\{A_k\}_{k=1}^{N_e}$ and $\{B_k\}_{k=1}^{N_e}$ by, for $1 \leq k \leq k_0$:

$$\begin{aligned}
\phi_k &= (1-\chi)\underline{\phi}_R(e_t,d,s_t) + \frac{e_k}{(1-\chi)(e_t+d_t)^\zeta} \\
A_k &= V(e_{k,t}) - \left((1-\chi)\underline{\phi}_R(e_t,d_t,s_t)(e_t+d_t)^\zeta + e_{k,t}\right)\frac{V(e_{k+1,t})-V(e_{k,t})}{e_{k+1,t}-e_{k,t}} \\
B_k &= \frac{V(e_{k+1,t})-V(e_{k,t})}{e_{k+1,t}-e_{k,t}}(1-\chi)\underline{\phi}_R(e_t,d,s_t)(e_t+d_t)^\zeta
\end{aligned}$$

and, for $k_0+1 \leq k \leq N_e-1$:

$$\begin{aligned}
\phi_k &= \chi\overline{\phi}_R(e_t,d_t,s_t) + (1-\chi)\underline{\phi}_R(e_t,d,s_t) + \frac{e_k}{(e_t+d_t)^\zeta} \\
A_k &= V(e_{k,t}) - \left(\chi\overline{\phi}_R(e_t,d_t,s_t) + (1-\chi)\underline{\phi}_R(e_t,d,s_t) + e_{k,t}\right)\frac{V(e_{k+1,t})-V(e_{k,t})}{e_{k+1,t}-e_{k,t}} \\
B_k &= \frac{V(e_{k+1,t})-V(e_{k,t})}{e_{k+1,t}-e_{k,t}}\left(\chi\overline{\phi}_R(e_t,d_t,s_t) + (1-\chi)\underline{\phi}_R(e_t,d,s_t)\right)(e_t+d_t)^\zeta
\end{aligned}$$

The two integrals are then approximated as:

$$\begin{aligned}
(INT-1)+(INT-2) = \;& \sum_{k=1}^{k_0}\int_{\phi_k}^{\phi_{k+1}}(A_k+B_k\phi)dF(\phi) \\
&+ \int_{\phi_{k_0}}^{\overline{\phi}_R(e_t,d_t,s_t)}(A_{k_0}+B_{k_0}\phi)dF(\phi) + \int_{\overline{\phi}_R(e_t,d_t,s_t)}^{\phi_{k_0+1}}(A_{k_0+1}+B_{k_0+1}\phi)dF(\phi) \\
&+ \sum_{k=k_0+1}^{N_e-1}\int_{\phi_k}^{\phi_{k+1}}(A_k+B_k\phi)dF(\phi).
\end{aligned}$$

This method works for $2 \leq k_0 \leq N_e-2$; the extensions to the cases $k_0=1$ and $k_0=N_e-1$ are straightforward.

There are a number of other cases to be covered, depending on whether $(d_t,s_t) \in \mathcal{S}_R(e_t)$ or $(d_t,s_t) \in \mathcal{S}_K(e_t)$, whether the implied contracts are riskless or not, and on the threshold at which the firm has sufficient cash on hand to issue dividends after the debt settlement stage. However, in all these cases the value function can be approximated using linear interpolation, as described above. The exact expressions for the value function, in each case, are available on request.

Computation of the invariant distribution The invariant distribution of firms over the set $[0,\bar{e}]$ is approximated using a discrete set of weights $\{w_i\}_{i=1}^{N_e}$, which represent the mass of firms with equity in the interval $\left]\frac{1}{2}(e_{i-1}+e_i);\frac{1}{2}(e_i+e_{i+1})\right]$ for $2 \leq i \leq N_e-1$, and the mass of firms in the interval $\left[0,\frac{1}{2}e_2\right]$ for $i=1$ and $\left]\frac{1}{2}(e_{N_e-1}+e_{N_e}),e_{N_e}\right]$ for $i=N_e$. The invariant distribution is the solution to the matrix equation $wP=w$, where $w=(w_1,...,w_{N_e})$ and M is a $(N_e \times N_e)$ matrix representing the transition kernel Q. Specifically, the entry $m_{i,j}$ is given by $m_{i,1}=N(e_i,\frac{1}{2}e_2)$, $m_{i,j}=N(e_i,\frac{1}{2}(e_j+e_{j+1}))-N(e_i,\frac{1}{2}(e_j+e_{j-1}))$ for

$2 \leq j \leq N_e - 1$, and $m_{i,j} = 1 - N(e_i, \frac{1}{2}e_{N_e-1}))$ for $j = N_e$. Note that computing the invariant distribution requires the entry scale e^e, which can be obtained using the solution to the firms' problem **(M-8)** along with the entry condition **(M-9)**.

Computation of the perfect foresight response of the economy Computing the perfect foresight response of the economy to aggregate shocks is straightforward, because there are no endogenous aggregate price paths to solve for, since the cost of funds of financial intermediaries, r, always equals $\frac{1}{\beta} - 1$. In a perfect foresight equilibrium, one must however ensure that firms' current decisions take into account future variation in the path of aggregate shocks; that is, the decision problem of each firm must be solved through backward iteration. I describe the algorithm used to compute the perfect foresight response when the aggregate shock affects the intermediation wedge γ. Let $\{\gamma^{(t)}\}_{t=0}^{+\infty}$ denote the path of the aggregate shock, and $\gamma^{(+\infty)} \equiv \lim_{t\to+\infty} \gamma^{(t)}$. γ_{-1} denotes the value of the intermediation wedge before the shock occurs.

1. Compute the long-run steady-state of the economy, that is, the steady-state of the economy when $\gamma = \gamma^{(+\infty)}$. Denote by $\bar{e}^{(+\infty)}$, $V^{(+\infty)}$, $(\hat{d}^{(+\infty)}, \hat{s}^{(+\infty)})$ and $w^{(+\infty)} = \left\{w_i^{(+\infty)}\right\}_{i=1}^{N_e}$ the upper bound on equity, the value function, policy functions and the discrete approximation to the steady-state distribution of firms in the long-run steady-state. Furthermore, let $\bar{e}^{(0)}$ and $w^{(0)}$ denote the upper bound on equity and the discrete weights approximating the invariant measure of firms when $\gamma = \gamma_{-1}$.

2. Fix $T > 0$. Let $\bar{e}^{(T)} \equiv \bar{e}^{(+\infty)}$, $V^{(T)} \equiv V^{(+\infty)}$, $(\hat{d}^{(T)}, \hat{s}^{(T)}) \equiv (\hat{d}^{(+\infty)}, \hat{s}^{(+\infty)})$. For $t = T-1, ..., 1$:

   (a) Start with the guess $\bar{e}^{(t)} = \bar{e}^{(t+1)}$.

   (b) Compute the frontiers of the feasible sets $\mathcal{S}_K^{(t)}(.;\gamma^{(t)})$ and $\mathcal{S}_R^{(t)}(.;\gamma^{(t)})$ for values of e_t on a discretized grid over $\left[0, \bar{e}^{(t)}\right]$.

   (c) Compute the value function $V^{(t)} = T^{(t)}V^{(t+1)}$ and the associated policy functions $(\hat{d}^{(t)}, \hat{s}^{(t)})$ on the discretized grid over $\left[0, \bar{e}^{(t)}\right]$. The index (t) indicates that the aggregate shock takes the value $\gamma^{(t)}$ in the formulation of the firm problem.

   (d) Check whether $(1-\eta)\beta\frac{\partial V^{(t)}}{\partial e}(\bar{e}^{(t)}) = 1$ to a sufficient degree of precision. If not, restart $(a) - (c)$ with a new guess for $\bar{e}^{(t)}$.

   (e) Compute the entry scale $e^{e,(t)}$ implied by $V^{(t)}$.

3. For $t = 0$, compute the policy functions $\left\{\hat{d}^{(t)}, \hat{s}^{(t)}\right\}_{t=0}^{T}$ on $\left[0, \bar{e}^{(0)}\right]$ that solve the maximization problem $V^{(0)} = T^{(0)}V^{(1)}$.

4. Next, given the sequence of upper bounds $\{\bar{e}^t\}_{t=1}^{T=0}$, policy functions $\left\{\hat{d}^{(t)}, \hat{s}^{(t)}\right\}_{t=0}^{T}$ and entry scales $\{e^{e,(t)}\}$ implied by the value functions, compute transition matrices $M_{t,t+1}$ for $0 \leq t \leq T-1$. As above,

the transition matrix is defined using the transition kernel $N^{(t)}(e_t, e_{t+1})$. This transition kernel takes the same expression as described above, and only depends on policy functions of the firm at time t, the equity bounds $\bar{e}^{(t)}$ and $\bar{e}^{(t+1)}$ and the entry scale $e^{e,(t+1)}$, as well as the current value of the aggregate shock $\gamma^{(t)}$.

5. The transition matrices $M_{t,t+1}$ map discrete weights $w^{(t)}$ on $\left[0, \bar{e}^{(t)}\right]$ to discrete weights on $w^{(t+1)}$ on $\left[0, \bar{e}^{(t+1)}\right]$ through $w^{(t+1)} = w^{(t)} M_{t,t+1}$. Use this to compute the evolution of the approximate firm measure for $t = 0, ..., T$, starting with $w^{(0)}$.

6. Check whether the implied $w^{(T)}$ is sufficiently close to $w^{(+\infty)}$. If not, increase T and repeat steps 1-5.

4 Additional results

4.1 Optimal debt structure in the static model

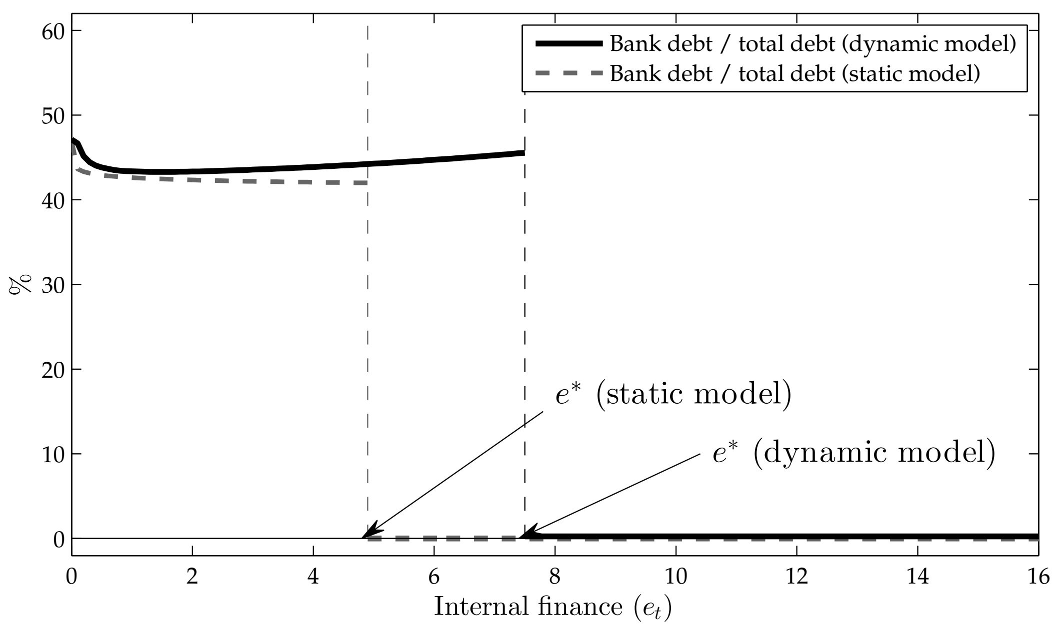


Figure 2: Optimal composition of debt in the static and dynamic models.

The main text mentions the fact that one of the drivers of firms' debt choice, in the model, is the concavity of their value function $V(e_t)$. Figure 2 illustrates this. It depicts the optimal debt structure of firms in the dynamic model (black line) and a static version of the model (grey line) with an identical calibration.[12] The static model can be thought of as the limiting case of the model studied in the main text, when $\beta = 0$ (provided that the cost of funds of financial intermediaries, r, is exogenously given). In that case, the firm has linear utility over final cash flows (after debt settlement). The resulting debt structure has

[12]This version of the model is identical to the static model I study in Crouzet (2013).

two differences with respect to the dynamic case: firms with mixed debt structures borrow less from banks; and the internal finance threshold at which firms switch to the market-only financing regime is smaller. Both of these differences indicate that the concavity of the valuation of cash flows, in the dynamic model, induces firms to increase their reliance on bank lending.

4.2 The relationship between asset size and debt composition

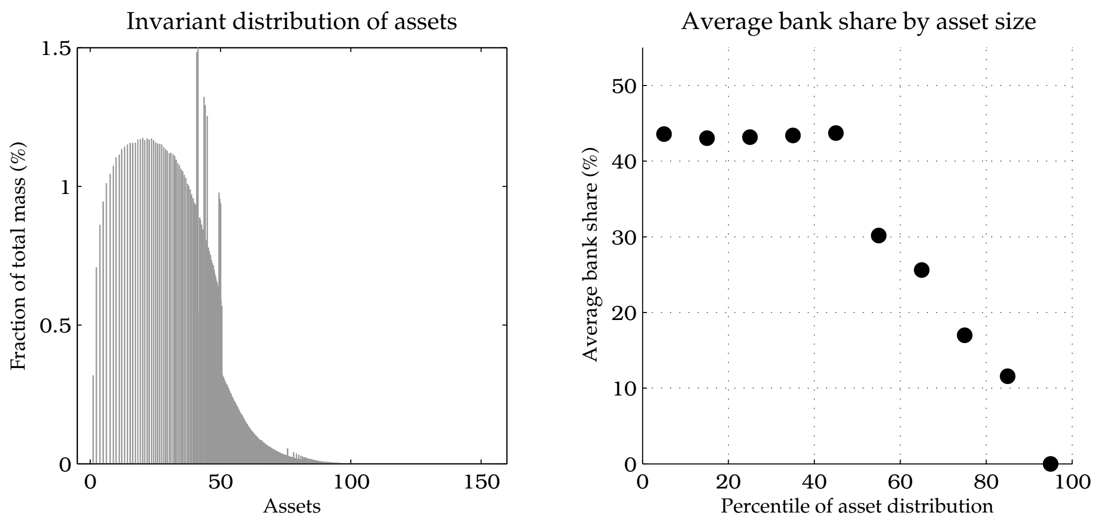


Figure 3: Asset size and debt structure in the steady-state of the model.

The left panel of figure 3 depicts the steady-state distribution of firms by asset size. Total assets of firms are given by $\hat{k}(e_t) = (e_t + d(\hat{e_t}))$. The implied distribution depicted on the left panel is obtained by by drawing large sample from the approximate distribution of firms $\{w_i\}_{i=1}^{N_e}$, and computing the value of $\hat{k}(e_{d,t})$ for each draw indexed by d. This distribution is positively skewed, reflecting the fact that the invariant measure of firms across internal finance levels e_t is also positively skewed, and additionally that total assets are a piecewise increasing function of internal finance. The "spikes" on the distribution occur because of the drop in total assets as firms switch from the mixed-finance to the market-finance regime. They coincide with asset levels that are optimal for both market-finance firms and mixed-financed firms. The right-hand side depicts the average bank share among firms within the successive quantiles of the asset distribution.[13] The bank share falls as one moves to the right of the invariant asset distribution, as is the case in the data reported in appendix.

[13]The first dot, for example, is the average bank share of firms in the bottom 5% of the asset size distributions.

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