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1 Some remarks about technology

The starting point of the Solow "models" is a general production function:

\[ F(K_t, L_t, A_t) \]

which describes the production possibilities of the economy as a whole. Production possibilities depend on physical capital, labour, and some technology shifter, \( A_t \). In most of what we do, we assume that this function has the some or all of the following properties:

1. \( F : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+ \) is twice continuously differentiable in its three arguments.

2. Production exhibits constant returns to scale:

\[ \forall \mu > 0, \forall x, y, \quad F(\mu x, \mu y, z) = \mu F(x, y, z) \]

3. Returns to inputs are positive and diminishing:

\[ F_1 > 0, F_2 > 0 \]
\[ F_{11} < 0, F_{22} < 0 \]

4. Inada conditions:

\[ \forall z, \]
\[ \forall y, \quad \lim_{x \to 0} F_1(x, y, z) = +\infty \]
\[ \forall x, \quad \lim_{y \to 0} F_2(x, y, z) = +\infty \]
\[ \forall y, \quad \lim_{x \to +\infty} F_1(x, y, z) = 0 \]
\[ \forall x, \quad \lim_{y \to +\infty} F_2(x, y, z) = 0 \]

5. Essentiality: \( \forall x, y, \quad F(x, 0, z) = F(0, y, z) = 0 \)

In this section, we'll focus on the first assumption, and study its implications, both for the production function and in terms of how well it corresponds to some features of the data. The particular features of the data that we will show are consistent with constant returns to scale are the three following of the "Kaldor facts":

1. Capital to output ratio is constant over time
2. Interest rate is constant
3. Distribution of income between capital and labour is constant

1.1 Euler’s theorem

These two useful results about homogeneity will be guiding much of our analysis of the Solow model with constant returns to scale.

Result 1 (Euler’s theorem). Let \( f : \mathbb{R}^M \to \mathbb{R} \) be a continuously differentiable, homogeneous of degree \( \alpha \) function, i.e. such that:

\[
\exists \alpha > 0 : \forall x \in \mathbb{R}^M, \forall \mu > 0, \quad f(\mu x) = \mu^\alpha f(x)
\]

Then,

\[
\sum_{m=1}^{M} \frac{\partial f}{\partial x_m}(x)x_m = \alpha f(x)
\]

Proof. By differentiability,

\[
\forall \mu > 0, \quad \frac{\partial}{\partial \mu} f(\mu x) = \sum_{m=1}^{M} \frac{\partial f}{\partial x_m}(\mu x)x_m
\]

By homogeneity of degree \( \alpha \),

\[
\forall \mu > 0, \quad \frac{\partial}{\partial \mu} f(\mu x) = \frac{\partial}{\partial \mu} \mu^\alpha f(x) = \alpha \mu^{\alpha-1} f(x)
\]

Taking \( \mu = 1 \) and equating the two lines above gives the result.

Result 2 (On a related note). Let \( f : \mathbb{R}^M \to \mathbb{R} \) be a continuously differentiable, homogeneous of degree \( \alpha \) function. Then, \( \forall m = 1, \ldots, M, x \to \frac{\partial f}{\partial x_m}(x) \) is homogeneous of degree \( \alpha - 1 \).

Proof. Differentiate both sides of the equality:

\[
f(\mu x) = \mu^\alpha f(x)
\]

with respect to \( x_m \).
Why do we care about these theorems? We can apply them to our constant returns to scale (in $K_t, L_t$) production function:

$$Y_t = F(K_t, L_t, A_t)$$

under: $M = 2$ and $\alpha = 1$. It gives us the following two results:

- From the second theorem, $F_1(K_t, L_t, A_t)$ and $F_2(K_t, L_t, A_t)$ are homogeneous of degree zero functions.

- From Euler's theorem:

$$Y_t = F(K_t, L_t, A_t) = K_tF_1(K_t, L_t, A_t) + L_tF_2(K_t, L_t, A_t)$$

Under cost minimization from a representative, price taking firm, we have have seen in class that:

$$R_t = F_1(K_t, L_t, A_t)$$

$$w_t = F_2(K_t, L_t, A_t)$$

so that:

$$Y_t = R_tK_t + w_tL_t$$

This says that in our constant returns to scale world, output is entirely exhausted by payments to factor prices.

### 1.2 Why do we know that technology must be labour-augmenting in the long run?

The first of the Kaldor "facts" mentioned initially will allow us to obtain the following result, which suggests that technology should, in the long-run, be "labour-augmenting".

**Result 3.** Assume that

$$F(K_t, L_t, A_t)$$

and the following:

- $F$ exhibits constant returns to $K$ and $L$.
- $L_t$ grows at the constant rate $n$.
- $\exists T$ s.t. $\forall t \geq T$,

$$\frac{\dot{Y}_t}{Y_t} = g_Y, \quad \frac{\dot{K}_t}{K_t} = g_K, \quad \frac{\dot{C}_t}{C_t} = g_c$$
• **Capital accumulates following:**

\[
\dot{K}_t = Y_t - C_t - \delta K_t
\]

Then, if the capital to output ratio is constant, we have that:

1. \( g_Y = g_K = g_C \)

2. \( \forall t \geq T, \exists F : \mathbb{R}^2 \rightarrow \mathbb{R}, \text{with constant returns to scale, such that:} \)

\[
Y_t = F(K_t, \tilde{A}_tL_t)
\]

\[
\text{and } \quad \frac{\dot{\tilde{A}}_t}{\tilde{A}_t} = g_Y - n
\]

**Proof.** During the recitation. \(\square\)

Note that this result only holds asymptotically, and moreover it’s only true if growth rates of input, output and consumption are constant in the long-run.

The intuition is as follows. In the long run, if output has a constant growth rate, with constant capital to output ratio, the growth rate of capital must be the same. Constant returns to scale in the production function imply that the growth rate of the labour input must also be the same. If this labour input grows exogenously, then either it just so happens that \( n \) is equal to \( g_Y \), or some change in technology has to make up for the difference between the two. This is why technology must be "labour-augmenting".

The type of production function \( Y_t = F(K_t, \tilde{A}_tL_t) \) is called "Harrod-neutral"; other flavors of technology, include "Solow-neutral" \( Y_t = F(A_tK_t, L_t) \) and "Hicks-neutral" \( Y_t = \tilde{A}_tF(K_t, L_t) \).
2 The Solow model in discrete time

2.1 Economic environment

- Discrete time: \( t = 0, 1, \ldots \)
- Closed economy
- Single good, used for consumption and investment.
- Neo-classical production function \( Y_t = F(K_t, A_t L_t) \) with Harrod-neutral technology.
- \( L_t \) is the population, while \( A_t \) is the level of labour-augmenting technology. Both grow at constant and exogenous rates, \( n \) and \( x \) respectively. We normalize their initial level to 1.
- Single representative household and single representative firm. Let’s not discuss this for now!

2.2 Competitive equilibrium

As in class, the representative firm takes factor prices as given and minimizes costs, so that:

\[
R_t = F_1(K_t, H_t)
\]

\[
\hat{w}_t = F_2(K_t, H_t)
\]

where \( H_t \) is effective labor demanded by the firm. The firm rebates any profits \( \Pi_t \) lump-sum to the household. The specifics of this does not matter, since constant returns to scale imply that these profits will be zero.

The household consumes goods produced, provides (effective) labour, rents out capital to the firm, and accumulates it between periods. Its budget constraint is:

\[
C_t + I_t = \Pi_t + \hat{w}_t H_t + R_t K_t = \hat{w}_t H_t + R_t K_t
\]

and the capital accumulation equation is:

\[
\tilde{K}_{t+1} - (1 - \delta) \tilde{K}_t = I_t
\]

Effective labour supplied by the household is given by: \( \bar{H}_t = A_t L_t \). The household takes the initial value of capital, \( K_0 \), as given.
The specific assumption of the Solow model is that the household follows the simple consumption rule:

\[ C_t = (1 - s)(\tilde{w}_t \tilde{H}_t + R_t \tilde{K}_t) \]

A competitive equilibrium of this economy can be defined as follows.

**Definition 2.1** (Competitive equilibrium of the Solow model). An equilibrium of the Solow model is an allocation \( \{C_t, K_t, I_t, L_t\}_{t \geq 0} \), and prices \( \{\tilde{w}_t, R_t\}_{t \geq 0} \) such that given the exogenous processes for \( A_t \) and \( L_t \):

- **Taking prices as given,** firms minimize input costs.
- **Households supply effective labour** \( \tilde{H}_t \) inelastically and follow the consumption rule, subject to their budget constraint and capital accumulation equation.
- **Capital and labour markets clear:** \( K_t = \tilde{K}_t \) and \( H_t = \tilde{H}_t \).

**Result 4** (Allocation in the competitive equilibrium of the Solow model). In the Solow model, the competitive equilibrium allocation follows:

\[ K_{t+1} - (1 - \delta)K_t = sF(K_t, A_t L_t) \]

*Proof.* Replace the market clearing conditions, the consumption rule and the capital accumulation equation into the budget constraint, and use the fact that \( F(K_t, A_t L_t) = \tilde{w}_t \tilde{H}_t + R_t \tilde{K}_t \).

### 2.3 The model in intensive units

Small cases denote "intensive units", that is, stuff per unit of effective labour:

\[
k_t \equiv \frac{K_t}{A_t L_T}, c_t = \frac{C_t}{A_t L_t}, y_t = \frac{Y_t}{A_t L_t}, i_t = \frac{I_t}{A_t L_t}
\]

We can rewrite our main equation as:

\[
(1 + n)(1 + x)k_{t+1} - (1 - \delta)k_t = s f(k_t)
\]

which we can (dodgily; details in class) approximate by:

\[
k_{t+1} = (1 - \delta - x - n)k_t + i_t
\]

This approximation is exact in the case of the continuous-time model. \( \delta + x + n \) is the "effective" rate of depreciation. In this recitation, take \( x = n = 0 \); all results would be identical, replacing \( \delta \) by \( \delta + x + n \).
2.4 Dynamics of the competitive equilibrium allocation

The dynamics of the model are governed by the transition equation:

$$k_{t+1} = T(k_t)$$

where

$$T(k_t) = sf(k_t) + (1 - \delta)k_t$$

This is a first-order autonomous difference equation in $k_t$, with initial condition $k_0 = K_0$. If we were able to solve it, we would also be able to study the dynamics of the whole of the competitive equilibrium allocation. Indeed:

$$K_t = A_tL_t k_t$$
$$Y_t = A_tL_t f(k_t)$$
$$C_t = (1 - s)A_tL_t f(k_t)$$
$$I_t = sA_tL_t f(k_t)$$
$$\dot{w}_t = f(k_t) - k_t f'(k_t)$$
$$R_t = f'(k_t)$$

Note also that the growth of capital per unit of effective labour in the competitive equilibrium allocation follows:

$$\frac{k_{t+1} - k_t}{k_t} \equiv g(k_t) = s \left(\frac{f(k_t)}{k_t} - \frac{\delta}{s}\right)$$

2.4.1 Steady-state and stability

**Result 5 (Steady-state).** A (non-trivial) steady-state of this economy is $k^* > 0$ such that:

$$T(k^*) = k^*$$

If $\delta \in ]0, 1[ \text{ and } s \in ]0, 1[\text{, then there exists a unique non-trivial steady-state of this economy. Furthermore, it is increasing in } s \text{ and decreasing in } \delta. \text{ Consumption } c^* \text{ in steady-state decreases with } \delta, \text{ but has a non-monotonic relationship with } s.$$

*Proof.* During the recitation.
The main stability result is identical to the discrete time case: whatever the initial value of capital, convergence to the steady-state always occurs, and it is monotonic. This property is called "global stability".

**Result 6** (Global stability). For any initial level of \( k_0 > 0 \), the equilibrium path of \( \{k_t\}_{t \geq 0} \) is such that:

\[
\lim_{t \to +\infty} k_t = k^*
\]

Furthermore, the transition is monotonic and the absolute value of the growth rate of capital \( g(k_t) \) decreases along the transition path.

**Proof.** During the recitation. \( \square \)

This result can be obtained graphically (see graphs in class).

### 2.4.2 Transitional dynamics

**Result 7** (Transitional dynamics). Imagine the economy starts at \( k_0 < k^* \). Then, along the transition path,

- \( y_t \) increases
- \( c_t \) increases
- \( R_t \) decreases, while \( \hat{w}_t \) increases

**Proof.** The two first results are simple. For the third one, remember that:

\[
R_t = f'(k_t) \\
\hat{w}_t = f(k_t) - k_t f'(k_t)
\]

First, as \( f \) is concave, \( R_t \) decreases along the transition path. Second, we have that:

\[
\frac{\partial}{\partial x} (f(x) - xf'(x)) = -xf''(x) > 0
\]

so that \( w_t \) increases along the transition path. \( \square \)

Intuitively, when \( k_0 < k^* \), we know the transition path involves an increase in capital per units of effective labour, \( k_t \). This means that capital becomes relatively more abundant than (effective) labour as an input, so the relative price \( \hat{w}_t / R_t \) increases.

The Euler theorem also gives us a nice way to find this results graphically (see graph in class).

[Graph of \( y = x \) and \( y = T(x) \)] [Graph of \( y = f(x)/x \) and \( y = \delta/s \)] [Graph of inputs prices during the transition]
3 Fun times with the Solow model

As Pr. Reis showed you in class, one nice thing about the Solow model is that it provides a framework simple enough to think easily about some important macro questions, while, at the same time, it’s rich enough to allow us to ask some quantitative questions.

In this section we’ll go over some super-simple examples of how one may use the Solow model to ask basic theoretical and empirical questions. We’ll look at whether the convergence rates implied by the model seem reasonable compared to the data. We’ll also see how the Solow model may provide a framework to think about poverty traps. Finally, we’ll quickly discuss the possibility of endogenous growth in the Solow model.

3.1 Convergence rates and the data

As we saw in class, the Solow model has some quantitative predictions about (conditional) convergence. Besides the sign of the correlation between income growth and the level of income (relative to steady-state), the model has specific predictions about the magnitude of this correlation. In this section, we’ll see one fairly straightforward way to test these predictions against the data, using convergence rates derived from the model.

Consider a continuous-time version of the Solow model with a Cobb-Douglas production function and labour-augmenting technology. The capital accumulation equation is:

$$\frac{\dot{k}_t}{k_t} = s \left( k_t^{\alpha - 1} - \frac{\delta + n + x}{s} \right)$$

First, because $y_t = k_t^\alpha$:

$$\frac{\dot{y}_t}{y_t} = \alpha \frac{\dot{k}_t}{k_t}$$

Second, rewrite $k_t = k^* e^{x_t}$, and take a first-order Taylor expansion of $x_t \to k^* e^{(\alpha - 1)x_t}$ around $x_t = 0$ to find:

$$\frac{\dot{k}_t}{k_t} \approx -(1 - \alpha)(\delta + n + x)x_t = -(1 - \alpha)(\delta + n + x)(\log(k_t) - \log(k^*))$$

Third, replace $\log(y_t) = \log(k_t^\alpha) = \alpha \log(k_t)$, with $y^* = (k^*)^\alpha$ in the equation above to get:

$$\frac{1}{\alpha} \frac{\dot{y}_t}{y_t} = \frac{\dot{k}_t}{k_t} \approx -(1 - \alpha)(\delta + n + x) \frac{1}{\alpha} (\log(y_t) - \log(y^*))$$
or:

\[
\frac{\dot{y}_t}{y_t} \approx -(1 - \alpha)(\delta + n + x)(\log(y_t) - \log(y^*))
\]

This is a first step towards obtaining convergence rates for output; however, we only observe GDP per capita, \(g_t = A_t y_t\). Defining:

\[g^*_t \equiv A_t y^*_t\]

and using the fact that technology grows at a constant rate, \(\frac{\dot{A}_t}{A_t} = g\), we get that:

\[
\frac{\dot{g}_t}{g_t} - g \approx -(1 - \alpha)(\delta + n + x)(\log(g_t) - \log(g^*_t))
\]

so that:

\[
\frac{d}{dt} \log(g_t/g^*_t) \approx -(1 - \alpha)(\delta + n + x)(\log(g_t/g^*_t))
\]

Assume that this approximation holds exactly. Letting \(X_t \equiv \log(g_t/g^*_t)\), this gives us a first order differential equation in \(X_t\), the solution to which is:

\[X_t = \log(g_0/y^*) \exp(-\lambda t)\]

where \(\lambda = (1 - \alpha)(\delta + n + x)\) is the (absolute value of the) rate of convergence.

We can use this simple remark to test the speed of conditional convergence in the data. Indeed, imagine that two countries are similar in the sense that they share the same parameters \(s, \delta, x, n, \alpha\), and therefore the same steady-state output per capita \(g^*_t\). This assumption may seem reasonable, for example, in the case of France and Spain.

Let \(X_{1,t}\) denote the solution to the equation above for the first country, and \(X_{2,t}\) denote that solution for the second country. The countries only differ by their initial income per capita ratio. The evolution of the log the ratio of the countries’ output per capita follows:

\[\log(g_{2,t}/g_{1,t}) = X_{2,t} - X_{1,t} = \log(g_{2,0}/g_{1,0}) \exp(-\lambda t)\]

where I used the fact that for the two "identical" countries, \(g^*_1, t = g^*_2, t\).

Suppose for example that initially \(g_{2,0} < g_{1,0}\); then we can compute how much time the model predicts it will take for the two countries to reach a point where \(g_{2,t} \geq x g_{1,t}\) for some \(x\) close to (but smaller than) 1. Formally, this \(t_m\) is given by:

\[t_m = \frac{1}{\lambda} \log \left( \frac{\log \left( \frac{g_{2,0}}{g_{1,0}} \right)}{\log(x)} \right)\]
To get an idea of whether this is a reasonable estimate of the actual speeds of convergence, take $\delta = 5\%$, $n = 2\%$ and $g = 2\%$. Along with the usual value of $\alpha = \frac{1}{3}$, this gives $\lambda = 6\%$.

Coming back to the case of France (country 1) and Spain (country 2), take time 0 to be 1969. A quick look at (for example) data made available by the BLS\(^1\) indicates that France was 22% richer (in per capita terms) than Spain at that time. In 1999, 30 years later, this ratio was 15%. $x = 1/1.15$ and $\frac{g_{t,0}}{g_{t,i}} = 1/1.22$ in the formula above for $t_m$ yields:

$$t_m \approx 2.5 \text{ years}$$

The model drastically overestimates the speed of convergence that we see in the data. We would need $\lambda \approx 0.5$ to get a 30 year convergence rate for these two countries. If you repeat this computation with other country pairs, you will find that for most countries the Solow model tends to overestimate the speed of convergence.

### 3.2 A model with a poverty trap

A recurring issue that poor countries seem to face is their inability to "start growing", despite repeated attempts to jumpstart growth, in particular through transfer of goods and technology from richer countries. In fact, some among the poorest countries - such as Zambia, for example - have even experienced negative rates of growth of output per capita over the last decades. This phenomenon is known as "poverty traps".

The Solow model provides us with a simple framework to guide understand why these poverty traps may exist. The basic idea is to think of a countries as having not one, but multiple locally stable steady-states, some of them with a lower level of income per capita than others. The fact that they are stable means that it is difficult to "escape" them; that is, large deviations from the steady-state level of capital are needed in order to shift the country away from the low steady-state. One simple policy implication of the model is that incremental aid will be of little help in extracting the country from the "bad" steady-state.

But how do we generate multiple non-trivial steady-states in the Solow model? Imagine the planner has access to two different technologies:

$$Y_t^A = AK_t^\alpha L_t^{1-\alpha}$$

$$Y_t^B = BK_t^\alpha L_t^{1-\alpha} - bL_t$$

where $A < B$. $b$ is a fixed cost (per unit of labor used).

The planner will use the more productive technology whenever it can obtain at least the same amount of output with it than with the less productive technology, that is, whenever:

$$BK_t^\alpha L_t^{1-\alpha} - bL_t \geq AK_t^\alpha L_t^{1-\alpha}$$

\(^1\)Data is available at: http://www.bls.gov/fls/.
or equivalently:

$$Bk_t^\alpha - b \geq Ak_t^\alpha$$

Therefore, there is a level $k^- = \left( \frac{b}{B-A} \right)^{1/\alpha}$ such that:

$$f(k_t) = \begin{cases} Ak_t^\alpha & \text{if } k_t \leq k^* \\ Bk_t^\alpha - b & \text{if } k_t > k^* \end{cases}$$

Since we have only changed the production possibilities in our baseline economy, the growth rate of capital still follows:

$$\frac{k_{t+1} - k_t}{k_t} = s \left( \frac{f(k_t)}{k_t} - \frac{\delta}{s} \right)$$

where now:

$$\frac{f(k_t)}{k_t} = \begin{cases} Ak_t^{\alpha-1} & \text{if } k_t \leq k^* \\ Bk_t^{\alpha-1} - b/k_t & \text{if } k_t > k^* \end{cases}$$

This function is still strictly decreasing for $k \leq k^-$ and $k \to +\infty$, but may be increasing on a range in $k \geq k^-$. In this case, there may be multiple equilibria.

In the picture, the parameters are assumed that there are three steady-states: $k^*_{\text{low}} < k^*_{\text{middle}} < k^*_{\text{high}}$. The middle steady-state is unstable, while the higher and lower steady-state (the poverty trap) are stable.

Besides the implication that incremental aid is most likely helpless, the model has interesting implications for the effects of technology transfer, in the form of higher growth rates of technological progress. Can you see what these implications are?

### 3.3 AK model

A key issue with Solow-type models is that in the absence of technological growth (that is, if $x = 0$), output per capita will eventually converge to a fixed level (and have a zero rate of growth). Without exogenous technological progress, diminishing returns always catch up, in the long run.

A class of models, which we will study in more depth in future classes, has tried to generate long-run growth without resorting to exogenous technological change. The basic idea, in these models, is to do away with decreasing returns to scale.
The simplest of such models follows from setting the production possibilities to be equal to:

\[ Y_t = AK_t \]

Output per capita is now:

\[ f(k_t) = Ak_t \]

Our equation for the growth rate of capital now becomes:

\[ \frac{k_{t+1} - k_t}{k_t} = s \left( \frac{f(k_t)}{k_t} - \delta \right) \]

\[ = sA - \delta \]

So long as \( sA > \delta \), the economy has positive long-run growth without any technological progress.

Although this example may seem simplistic, models in which capital has both human and physical components will often have similar properties along their growth path (sustained growth even in the absence of technological progress).

### 4 The Mankiw-Romer-Weil model of human capital

This model extends the basic Solow model we studied previously by incorporating human capital into the production function. Human capital is accumulated from saving output, just as physical capital. Think of this as bypassing current consumption in order to increase future returns on labour; that is, basically, going to school. (This interpretation will motivate MKR's use of schooling rates as proxies for saving rates in human capital in their empirical approach).

#### 4.1 Model equations

Assume output is produced by a homogeneous of degree 1 production function with positive and diminishing returns, \( Y_t = F(K_t, H_t, A_tL_t) \). It takes as inputs physical capital \( K_t \), human capital \( H_t \), and effective labour \( A_tL_t \). Further, assume that saving rates for human and physical capital are \( s_h \) and \( s_k \), respectively, while depreciation rates are \( \delta_h \) and \( \delta_k \), respectively. Of course, \( s_h + s_k < 1 \).

The two types of capital accumulate following:

\[ \dot{K}_t = (1 - \delta_k)K_t + I_{k,t} \]
\[ \dot{H}_t = (1 - \delta_h)H_t + I_{h,t} \]

As previously, we can put the system in stationary form by normalizing all variables by \( A_t L_t \). The stationary production function now takes the form:

\[ y_t = f(k_t, h_t) \]

Using the normalized form of the capital accumulation equations, and the constant savings rate in each type of capital, we get the following dynamic system (in stationary form):

\[
\begin{align*}
\dot{k}_t &= s_k \left( \frac{f(k_t, h_t)}{k_t} - \frac{\delta_k + n + x}{s_k} \right) \\
\dot{h}_t &= s_h \left( \frac{f(k_t, h_t)}{h_t} - \frac{\delta_h + n + x}{s_h} \right)
\end{align*}
\]

### 4.2 Existence, uniqueness and stability of the steady-state

As in the one-sector case, we would like to study the solution to this system of autonomous difference equations, and in particular, determine whether a steady-state exists, is unique, and if so, whether it also stable. The following result tells us that if we add in Inada-type conditions, this system has a unique steady-state, and this steady-state is stable.

**Result 8** (Existence, uniqueness and global stability of the steady-state in MKR). *Assume that the following conditions hold:*

\[
\begin{align*}
\lim_{k \to +\infty} f_1(k, h) &= 0 \\
\lim_{h \to +\infty} f_2(k, h) &= 0 \\
\lim_{k \to 0} f_1(k, h) &= +\infty \\
\lim_{h \to 0} f_2(k, h) &= +\infty
\end{align*}
\]

*Then, the system above has a unique steady-state, and this steady-state is globally stable.*

**Proof.** During the recitation (if time permits)

The stability result can be easily seen graphically with a phase diagram in \((k, h)\) space. The particular configuration depicted in the graph (upward slopes at intersection; unique intersection) come respectively from decreasing marginal returns to inputs and the Inada-type conditions. [Phase diagram in (h,k) space]
4.3 Steady-state with a Cobb-Douglas production technology

MKR uses a Cobb-Douglas form for the production function:

\[ Y_t = K_t^\alpha H_t^\beta (A_tL_t)^{1-\alpha-\beta} \]

which leads to:

\[ y_t = k_t^\alpha h_t^\beta \]

With this function form in hand, we can solve analytically for the steady-state of the system:

\[ k^* = \left( \frac{s_h}{n + g + \delta_h} \right)^\beta \left( \frac{s_k}{n + g + \delta_k} \right)^{1-\beta} \frac{1}{1-\alpha-\beta} \]

\[ h^* = \left( \frac{s_k}{n + g + \delta_k} \right)^\alpha \left( \frac{s_h}{n + g + \delta_h} \right)^{1-\alpha} \frac{1}{1-\alpha-\beta} \]

\[ y^* = \left( \frac{s_k}{n + g + \delta_k} \right)^{\frac{\alpha}{1-\alpha-\beta}} \left( \frac{s_h}{n + g + \delta_h} \right)^{\frac{\beta}{1-\alpha-\beta}} \]

4.4 Empirical successes...

In order to bring this model to bear on the data, MKR assume that the world is composed of \( j = 1...J \) isolated countries, which differ through their human and capital savings rates \( s_{j,h} \) and \( s_{j,k} \), and their population growth rates, \( n_j \). They assume that \( \delta \) and \( g \) are identical across countries.

Furthermore, they assume that all countries are at their steady-state, so that their steady-state level of income per capita follows:

\[ y^*_{j,t} = A_{j,t} \exp(y^*_j) \]

Then, the steady-state solution:

\[ \log(y^*_{j,t}) = \log(A_j) + gt + \frac{\alpha}{1-\beta-\alpha} \log \left( \frac{s_{k,j}}{n_j + g + \delta} \right) + \frac{\beta}{1-\beta-\alpha} \log \left( \frac{s_{h,j}}{n_j + g + \delta} \right) \]

In order to be able to estimate this model by OLS, they finally make the following crucial assumption:

\( A_j = A \exp(\epsilon_j) \), where \( \epsilon_j \) is white noise, independent from all other variables on the right hand side.
Savings rate in human capital are proxied by the percentage of the working age population enrolled in school. The rest of the right hand side variable are know, except $\delta + g$, which is fixed to be equal to 5% in all countries. The key estimation results are given in the two following tables.

[Tables from MKR to be added]

4.5  ... Well, kind of.

Three main issues with their estimation approach:

- Omitted variable bias
- Reverse causality
- Micro evidence suggests that returns to human capital are smaller than implied by $\beta \approx \frac{1}{3}$

[To be completed]