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1 Asset prices in a 2-period Arrow-Debreu economy

This exercise is taken from the certification exam of 2008. It’s both good training for the midterm and the final, as well as a clear illustration of why, so long as asset markets are complete, the specific asset structure that achieves completeness is irrelevant.

The economy’s characteristics are as follows:

• Endowment economy with a single good (no production).

• Two periods: $t = 0, 1$.

• Two states in the second period: $s_h$ (occurs with probability $\pi(s_h) = 1/2$), $s_l$ (occurs with probability $\pi(s_l) = 1/2$).

• Two agents: Type $A$, earning a constant amount $y^A = e$ each period ("workers"); type $B$, who own a "tree" that earns them $y^B(s_0) = 0$ in the first period, and, in the second period $y^B(s_h) = 2$ if the high state occurs, and $y^B(s_l) = 0$ if the low state occurs ("capitalists").

• Both types of agents ($i = A, B$) choose consumption to maximize expected discounted utility:

$$ u(c^i(s_0)) + \beta \left( \pi(s_l) u(c^i(s_l)) + \pi(s_h) u(c^i(s_h)) \right) $$

where:

$$ u(c) = 10c - \frac{1}{2}c^2 $$

1.1 Market structure 1: Arrow-Debreu securities

In this section we consider the following financial market structure: in period 0, agents can trade Arrow-Debreu assets. In our economy, there are two Arrow-Debreu assets: one that delivers one unit of the good if $s_h$ state realizes tomorrow (price $q(s_h)$), the other that delivers one unit of the good if $s_l$ realizes tomorrow (price $q(s_l)$).

Note that in this economy, since there is only one period, Arrow-Debreu securities and Arrow securities are identical.
### 1.1.1 Competitive equilibrium

Agent \( i = A, B \) maximizes the expected discounted value of future profits taking the prices of Arrow-Debreu securities as given, subject to his budget constraints:

\[
\max_{c^i(s_0), c^i(s_l), c^i(s_h), a^i(s_l), a^i(s_h)} \quad u(c^i(s_0)) + \frac{\beta}{2} \left( u(c^i(s_l)) + u(c^i(s_h)) \right)
\]

subject to:

\[
\begin{align*}
    c^i(s_0) + q(s_l) a^i(s_l) + q(s_h) a^i(s_h) &\leq y^i(s_0) \quad (\lambda_{i,0}) \\
    c^i(s_h) + a^i(s_h) &\leq y^i(s_h) \quad (\lambda_{i,h}) \\
    c^i(s_l) + a^i(s_l) &\leq y^i(s_l) \quad (\lambda_{i,l})
\end{align*}
\]

A competitive equilibrium of our economy is a list of prices \((q(s_l), q(s_h))\), asset holdings \((a^A(s_l), a^A(s_h), a^B(s_l), a^B(s_h))\), and consumptions \((c^A(s_0), c^A(s_l), c^A(s_h), c^B(s_0), c^B(s_l), c^B(s_h))\), such that:

1. For each agent, given prices, asset holdings and consumption solve the problem above
2. Asset markets clear in period 0:
   \[
   \begin{align*}
   a^A(s_l) + a^B(s_l) &= 0 \\
   a^A(s_h) + a^B(s_h) &= 0
   \end{align*}
   \]

You can easily check that Walras’ law holds here: because utility is strictly increasing in consumption in each period and state, the budget constraint hold with equality. Given this, asset market clearing implies goods market clearing by summing the budget constraints across agents for a given date and state.

Also, note that because we are in a finite economy, the natural debt limit and the No-Ponzi scheme condition are "irrelevant": we simply assume that there is no possible borrowing or saving in period 1.

### 1.1.2 Euler equations

The problem above consists in maximizing a strictly concave function on a convex constraint set. We can use a Lagrangian to solve this problem. To be absolutely explicit, the Lagrangian one should form here is:

\[
\mathcal{L} = u(c^i(s_0)) + \lambda_{i,0} (y^i(s_0) - c^i(s_0) - q(s_l) a^i(s_l) - q(s_h) a^i(s_h)) \\
+ \frac{\beta}{2} (u(c^i(s_l)) + \lambda_{i,l} (y^i(s_l) + a^i(s_l) - c^i(s_l))) \\
+ \frac{\beta}{2} (u(c^i(s_h)) + \lambda_{i,h} (y^i(s_h) + a^i(s_h) - c^i(s_h)))
\]
It doesn’t matter whether you put the state-by-state constraints "inside" our "outside" the expectation, but it turns out to be more convenient for the computations.

Solutions to the maximization problem of agent $i$ are characterized by the following sufficient first order conditions:

$$\frac{\partial L}{\partial c_i(s_0)} = 0 : \ 10 - c_i(s_0) = \lambda_{i,s_0}$$

$$\frac{\partial L}{\partial c_i(s_h)} = 0 : \ 10 - c_i(s_h) = \lambda_{i,s_h}$$

$$\frac{\partial L}{\partial c_i(s_l)} = 0 : \ 10 - c_i(s_l) = \lambda_{i,s_l}$$

$$\frac{\partial L}{\partial a_i(s_h)} = 0 : \ q(s_h)\lambda_{i,s_0} = \frac{\beta}{2}\lambda_{i,s_h}$$

$$\frac{\partial L}{\partial a_i(s_l)} = 0 : \ q(s_l)\lambda_{i,s_0} = \frac{\beta}{2}\lambda_{i,s_l}$$

along with the complementary/slackness conditions:

$$\lambda_{i,0}\frac{\partial L}{\partial \lambda_{i,0}} = 0 \ \text{and} \ \lambda_{i,0} \geq 0$$

$$\lambda_{i,l}\frac{\partial L}{\partial \lambda_{i,l}} = 0 \ \text{and} \ \lambda_{i,l} \geq 0$$

$$\lambda_{i,h}\frac{\partial L}{\partial \lambda_{i,h}} = 0 \ \text{and} \ \lambda_{i,h} \geq 0$$

In general, you don’t need to write the complementary/slackness conditions because the first order conditions imply that $\lambda > 0$ for every constraint of the problem (because utility is strictly increasing), so that the complementary/slackness conditions boil down to the budget constraints holding with equality. In our case, we will assume that consumption is strictly smaller than 10 (we can show that the solution we will find yield higher utility than consuming ten units in every state), so that we can also forget about the complementary/slackness conditions and replace them with budget constraints holding with equality.

Replacing out the Lagrange multipliers, we get the two following Euler equations for each agent:

$$10 - c_i(s_0) = \beta E \frac{1}{q(s_h)}(10 - c_i(s_h))$$

$$10 - c_i(s_0) = \beta E \frac{1}{q(s_l)}(10 - c_i(s_l))$$

Here, you might be a bit thrown off by the fact that you expected an Euler equation of the form:

$$u'(c_0) = \beta E R u'(c_1)$$

where $R$ is the gross return on the asset between periods 0 and 1, which is a random variable. Actually, this is exactly the form of the equations above. For example, for the low-state Arrow-Debreu asset, $R(s_l) = \frac{1}{q(s_l)}$ and
\( R(s_h) = 0 \), so that:

\[
\beta \mathbb{E} R u'(c_1) = \beta \left( \frac{1}{2} \frac{1}{q(s_l)} (10 - c_i(s_l)) + \frac{1}{2} 0. (10 - c_i(s_h)) \right)
\]

which is exactly the Euler equation for the Arrow Debreu asset that we found above. The pricing formula:

\[
u'(c_0) = \beta \mathbb{E} R u'(c_1)
\]

is very general and holds for any asset with random return \( R \), as we will see in the the class about asset pricing.

### 1.1.3 Equilibrium asset prices

With quadratic utility, it turns out that solving for the price of assets is really easy. The goods market clearing conditions, which resulted from the asset market clearing conditions and the budget constraints, are:

\[
c_A(s_0) + c_B(s_0) = y
\]

\[
c_A(s_l) + c_B(s_l) = y
\]

\[
c_A(s_h) + c_B(s_h) = y + 2
\]

By taking sums of the Euler equation for each asset across agents and replacing out the goods market clearing conditions, we get that:

\[
20 - y = \beta \frac{1}{2q(s_h)} (20 - (y + 2))
\]

\[
20 - y = \beta \frac{1}{2q(s_l)} (20 - y)
\]

which implies that:

\[
q_h = \frac{\beta}{2} \frac{20 - (y + 2)}{20 - y}
\]

\[
q_l = \frac{\beta}{2}
\]

As we would have expected, \( q_h < q_l \): transferring resources into the low aggregate state is more costly.
1.2 Market structure 2: a bond and a share

Now, we consider an alternative market structure: instead of being able to trade Arrow-Debreu assets, the two agent types have access to one bond and one share of the tree owned by the capitalists. The bond costs \( p_b \) in period 0 pays off 1 regardless of the state that occurs. Buying a share \( x \) of the tree costs \( p_s x \) in the first period and yields \( 2x \) in the high state, and 0 in the low state. Since there is initially only one tree, \( 0 \leq x \leq 1 \).

1.2.1 Budget constraints

Let \( x_i \) denote the share holdings of agent \( i \) and \( b_i \) denote their bond holdings. The budget constraints for the capitalist are now:

\[
\begin{align*}
    c_B(s_0) + p_b b_B + p_e x_B & \leq p_e \\
    c_B(s_h) & \leq b_B + 2x_B \\
    c_B(s_l) & \leq b_B
\end{align*}
\]

Note that since the capitalist initially owns the tree, the initial value of the tree enters his budget constraint. Once the state is realized in the first period, the tree becomes worthless, so the share has no resale value. The budget constraints of the worker as similar, only without the initial tree and with their constant income:

\[
\begin{align*}
    c_A(s_0) + p_b b_A + p_e x_A & \leq y \\
    c_A(s_h) & \leq b_A + 2x_A + y \\
    c_A(s_l) & \leq b_A + y
\end{align*}
\]

1.2.2 Competitive equilibrium

A competitive equilibrium of this economy is defined exactly as in the previous section, but now the asset market clearing condition becomes:

\[
\begin{align*}
    x_A + x_B & = 1 \\
    b_A + b_B & = 0
\end{align*}
\]

Bond are in zero net supply in this economy, while the shares of the tree owned by the two agents have to sum up to 1.
1.2.3 Euler equations

Again, for each agent, we have one Euler equation per asset. Through similar steps as in the previous section, we can check that the Euler equations have the form $u'(c_0) = \beta E_R u'(c_1)$. For agent $i$, the two Euler equations are:

\[
10 - c_i(s_0) = \beta \left( \frac{1}{2} p_b (10 - c_i(s_h)) + \frac{1}{2} p_s (10 - c_i(s_l)) \right) = \frac{\beta}{2} \left( \frac{2}{p_b} (10 - c_i(s_h)) \right)
\]
\[
10 - c_i(s_0) = \beta \left( \frac{1}{2} p_b (10 - c_i(s_h)) + \frac{1}{2} p_s (10 - c_i(s_l)) \right)
\]

1.2.4 Equilibrium asset prices

Again, we can add up the Euler equations for each asset across agents and use the goods market clearing conditions to find:

\[
p_b = \frac{\beta}{2} \left( \frac{20 - (y + 2)}{20 - y} + \frac{20 - y}{20 - y} \right)
\]
\[
p_s = \frac{\beta}{2} \left( 2 \frac{20 - (y + 2)}{20 - y} \right)
\]

It’s easy to see that:

\[
p_b = q(s_l) + q(s_h)
\]
\[
p_s = 2 q(s_h)
\]

The intuition behind this result is straightforward. Buying one unit of each of the two Arrow-Debreu assets, the agents can get an asset that has the same payoff in each state of the world as the bond. With identical state-contingent payoffs, the prices of the combined AD asset and the bond should also be identical. The same reasoning holds for the shares.

1.2.5 Equilibrium consumption

We can take the previous intuition in the other direction. Specifically, we can replicate the Arrow-Debreu securities using our bond and our shares:

- The AD asset for the high state is replicated by buying $\frac{1}{2}$ shares in the tree.
• The AD asset for the low state is replicated by buying one unit of the riskless bond and selling $\frac{1}{2}$ shares in the tree.

By "replicating", what is meant is creating an asset that has the same state-contingent payoff as the AD securities. The second asset, for example, pays off $0 \cdot \frac{1}{2} + 1 = 1$ in the low state and $2 \cdot \frac{1}{2} + 1 = 0$ in the high state, just like the AD asset.

Thus, the two asset structures allow us to transfer resources across dates and states in exactly the same way. Since under AD assets, markets were complete (by definition), markets are also complete with bond-equity trading. The resulting consumption allocations will be identical under the two market structures (a fact that you can check by using the prices we solved for).

2 Some welfare computations

In this section, we’re going to go over the question of the welfare costs of fluctuations again. The goal is to get some practice with the log-normal distribution, as well as assessing how important the following two assumptions that Lucas made are in obtaining his striking result:

• What if fluctuations of consumption around its trend are not $i.i.d.$, but persistent?

• What if there is no representative agent?

Throughout, we will be using the following super important property of log-normal random variables. A random variable $X$ is said to be log-normally distributed if:

$$\ln(X) \sim N(\mu, \sigma^2)$$

In this case,

$$\mathbb{E}(X) = \exp\left(\mu + \frac{1}{2}\sigma^2\right)$$

Also, throughout, utility is CRRA:

$$u(C) = \frac{C^{1-\gamma}}{1-\gamma}$$
2.1 The welfare cost of persistent fluctuations

All computations are from the perspective of some initial date 0. In particular, all expectations are conditional on information available at date 0.

Assume consumption follows:

\[ C_t = C_0 \exp (gt) \exp (\hat{c}_t) \]

where:

\[ \hat{c}_t = \eta c_{t-1} + \epsilon_t \]
\[ \epsilon_t \sim N(0, \sigma^2) \]
\[ \hat{c}_0 = 0 \]

Note that:

\[ \hat{c}_t \sim N(0, \text{var}_0(\hat{c}_t)) \]
\[ \text{var}_0(\hat{c}_t) = \sigma^2 \frac{1 - \eta^2 t}{1 - \eta^2} \]

Now, for any \( \gamma > 0 \),

\[ \ln(C_t^{1-\gamma}) \sim N \left( \ln(C_0^{1-\gamma}) + g(1 - \gamma)t, (1 - \gamma)^2 \text{var}_0(\hat{c}_t) \right) \]

so that:

\[ \mathbb{E}_0 C_t^{1-\gamma} = C_0^{1-\gamma} \exp \left( g(1 - \gamma)t + \frac{1}{2} (1 - \gamma)^2 \text{var}_0(\hat{c}_t) \right) \]

On the other hand,

\[ (\mathbb{E}_0 C_t)^{1-\gamma} = C_0^{1-\gamma} \exp \left( g(1 - \gamma)t + \frac{1}{2} (1 - \gamma) \text{var}_0(\hat{c}_t) \right) \]

So that:

\[ \mathbb{E}_0 C_t^{1-\gamma} = (\mathbb{E}_0 C_t)^{1-\gamma} \exp \left( \frac{1}{2} \gamma(\gamma - 1) \text{var}_0(\hat{c}_t) \right) \]

Remember that we define the welfare costs of fluctuations in terms of average consumption as the scalar \( \lambda \) that solves:

\[ \mathbb{E}_0 \left[ \sum_{t=0}^{+\infty} \exp(-\beta t)u((1 + \lambda)C_t) \right] = \sum_{t=0}^{+\infty} \exp(-\beta t)u(C_0 \exp(gt)) \]
After simplifying by $\frac{1}{1-\gamma}$, defining the effective discount rate $\rho = \beta - g(1-\gamma)$, and some manipulation (in class), we get that:

$$\ln(1 + \lambda) = \frac{1}{\gamma - 1} \ln(1 - \exp(-\rho\vartheta)) \left[ \sum_{t=0}^{+\infty} \exp(-\rho t) \exp \left( \frac{1}{2} \gamma(\gamma - 1) \vartheta \right) \right]$$

Now, the sum to the right hand side is not directly computable. Instead, we want to get some approximation of its value. Since we know that $\sigma^2$ is small, we can carry out a first-order Taylor expansion of the right hand side around $\sigma^2 = 0$. This leads to:

$$\ln(1 + \lambda) = \frac{1}{\gamma - 1} \frac{1}{2} \sigma^2 \gamma(\gamma - 1) \exp(-\rho) \frac{1}{1 - \exp(-\rho) \eta^2} + o(\sigma^2)$$

Further simplifying and approximating,

$$\lambda \approx \frac{1}{2} \sigma^2 \gamma \frac{1}{\rho + 1 - \eta^2}$$

Lucas’ costs of fluctuations were:

$$\lambda \approx \frac{1}{2} \sigma^2 \gamma \frac{1}{\rho + 1}$$

This calculation emphasizes that if the welfare costs of fluctuations may be large if fluctuations are persistent, that is, if $\eta$ is close enough to 1.

To get a sense of how large $\eta$ might be in the real data, a back of the envelope calculation (which turns out to be not totally inaccurate) is to think of the duration of deviations of consumption from its trend. These are shaped by the parameter $\eta$: the higher the $\eta$, the longer a unit shock to $\hat{c}_t$ takes to dissipate, and so the longer the consumption process takes to return to its trend. One can think of such a deviation as an expansion, or a recession, if the shock is negative. Say we think that the duration of an expansion is 3 years, that is, it takes roughly 3 years for consumption after a boom to return within, say, 30% of its trend value. The persistence $\eta$ corresponding to duration of deviation is:

$$\eta^3 = 0.3 \Leftrightarrow \ln(\eta) = \frac{\ln(0.3)}{3}$$

This gives $\eta \approx 0.85$, which implies welfare costs of fluctuations roughly 3.3 times bigger than in Lucas’ computation (taking $\rho \approx 0.02$). This is still extremely small in consumption terms: remember that Lucas’ estimates pointed to $\lambda \approx 0.01\%$. 


2.2 The welfare cost of consumption dispersion in the cross-section

Motivated by the empirical evidence on the increase in cross-sectional dispersion of consumption, let us assume that consumption for agent $i$ is described by:

$$C_{i,t} = C_0 \exp (gt + \hat{c}_{i,t})$$

$$\hat{c}_{i,t} \sim N(0, \sigma_i^2)$$

We will take a slightly different route than the Lucas computation. Using the "instantaneous" welfare measure:

$$W(t, \sigma_i^2) = \mathbb{E}[u(C_{i,t})]$$

what is the change in welfare implied by an increase in cross-sectional dispersion of consumption (an increase in consumption inequality) between two dates, $t$ and $t + T$?

Computations along the lines of the previous subsection give:

$$W(t + T, \sigma_{i+T}^2) - W(t, \sigma_i^2) \approx \exp \left( (1 - \gamma) \log(c_0) + (1 - \gamma)gt \right) \left[ gT + \frac{1}{2} (1 - \gamma)(\sigma_{i+T}^2 - \sigma_i^2) \right]$$

In the above expression, the first term in brackets reflects the (first-order) effect of the increase in mean consumption. The second term reflects the decrease in welfare (when $\gamma < 1$) that results from the increase in consumption inequality.

From the graphs above, we can see that the increase in consumption inequality is in the range $0.05 - 0.1$, which points to large welfare effects of consumption dispersion, of the order of $2\% - 4\%$ of mean consumption.

Note that in this computation is not the same as Lucas’, since it is not looking at the costs of fluctuations, but at the cost of cross-sectional dispersion in consumption. A more thorough analysis of the question of the welfare costs of cyclical fluctuations in a non-representative agent setup would require us to add a layer of uncertainty in mean income to the previous computation. This is done in Constantinides and Duffie (1996) or Krebs (2007). A key assumption is that the volatility of consumption must increase during recessions in order to get large costs of business cycles. The data seems to suggest that income and consumption dispersion are indeed counter-cyclical. The magnitude of the welfare effects are much larger than in Lucas’ computation and can reach $10\%$ - not that far from the simple assessment we just made.
G6215.001 - Recitation 5: Asset prices in an Arrow-Debreu economy; welfare computations
Equivalized consumption expenditures = nondurables, services, small durables and estimated flow from vehicles and housing.