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1 Simple questions from past midterms

1.1 Question 1

Question: A consumer wishes to maximize the expected discounted sum of utility from consumption, using funds from a stochastic stream of income that has a lower bound. Only saving is allowed, at interest rate R. The consumer has quadratic utility. If today he learns that the variance of his income has gone up, what happens to his expected consumption growth? How does your answer depend on \( y_0 \) and \( R \)?

Answer: Under quadratic utility, and if the no-borrowing constraint is not binding, consumption follows a martingale, assuming \( \beta = R^{-1} \):

\[
    c_t = \mathbb{E}_t c_{t+1}
\]

Thus, nothing should happen to consumption growth - consumption growth depends only on income variance if marginal utility is convex, that is, if there is a precautionary savings motive. The answer does not depend on \( y_0 \). It does not depend on \( R \): if \( R \neq \beta^{-1} \), expected consumption growth is unchanged since it is equal to:

\[
    (1 - \beta R) c_t
\]

which is independent of the variance of consumption tomorrow.

1.2 Question 2

Question: An economist is studying the impact of the stock market on consumption. He runs three different regressions:

- \( C_{t+1} - C_t = \alpha + a S_{t+1} \)
- \( C_{t+1} - C_t = \alpha + b S_t \)
- \( C_{t+1} - C_t = \alpha + c S_t^2 \)

He finds \( a > 0 \), \( b = 0 \) and \( c > 0 \). In a model of intertemporal consumption with a fixed real interest rate, what assumption would justify these results?
**Answer**: Consider a model with quadratic utility, then consumption follows:

\[ C_{t+1} = C_t + \epsilon_{t+1} \]

\[ \mathbb{E}_t \epsilon_{t+1} = 0 \]

The fact that consumption changes only depend on information known at time \( t + 1 \), but unknown at time \( t \), accounts for the two first findings. Since \( \text{var}(S_t)^2 \) is known at time \( t \), the model does not account for the third observation. One possibility is to add a precautionary savings motive. If income is positively correlated with stock market returns, then (iii) indicates that consumption growth is positively correlated with the variance of income. This statement is consistent with a model where marginal utility is convex.

### 1.3 Question 3

**Question**: An economy has one representative agent and three assets. Asset A has the same return next period regardless of the state of the world. The return on asset B has positive variance and is uncorrelated with the marginal utility of consumption. The return on asset C has positive variance and is negatively correlated with the marginal utility of consumption. Do you expect the return on asset A to be higher/ lower / the same as that of asset B? Do you expect the return of asset B to be higher / lower / the same as that of asset C? Explain.

**Answer**: Assets that pay off in states of the world where marginal utility of consumption are desirable to hold for the representative agent, since they provide insurance. Since the representative agent will want to purchase more of this asset than the riskless asset, in equilibrium, the price of this asset is should be higher than the price of a riskless asset. Correspondingly, the return on asset A should be higher than the return on asset C. Since the return on asset B is uncorrelated with marginal utility, it provides no additional insurance to the representative agent with respect to the riskless asset, and the return on the asset should therefore be identical to that of the riskless asset.

### 1.4 Question 4

**Question**: In an Aiyagari-type economy, an innovation in financial markets leads to borrowing constraint becoming looser. What happens to the equilibrium real interest rate?
**Answer**: In Aiyagari economies, borrowing constraints force the agents to accumulate a buffer stock of wealth. In equilibrium, the large demand for assets drives up the interest rate. If the borrowing constraint is made looser, buffer wealth accumulation decreases, asset demand goes down, and the interest rate goes up. (See graph in class).

### 2 Some closed-form solutions to dynamic programming problems

In this section, we’re going to practice solving dynamic programming problems, with two examples where solutions can be found in closed form.

#### 2.1 The Brock-Mirman model of stochastic growth

Consider the problem of a social planner who solves:

$$
\max_{C_t, K_{t+1}} \mathbb{E}_0 \sum_{t=0}^{+\infty} \beta^t \ln(C_t) \\
\text{s.t. } C_t + K_{t+1} \leq \exp(z_t) K_t^\alpha \\
K_t, C_t \geq 0 \\
\lim_{j \to +\infty} \beta^j \mathbb{E}_t \lambda_{t+j+1} K_{t+j+1} \geq 0
$$

The last condition is our usual transversality condition.

Two key elements here: the rate of depreciation of capital is $\delta = 1$; and utility is logarithmic, so that the marginal utility of income is the inverse of consumption. These two conditions will allow us to obtain a closed form solution to the problem. Setup the problem in Bellman form:

$$
V(K_t, \exp(z_t)) = \max_{C_t} \ln(C_t) + \beta \mathbb{E}_t V(K_t, \exp(z_t)) \\
\text{s.t. } C_t + K_{t+1} \leq \exp(z_t) K_t^\alpha \\
K_t, C_t \geq 0 \\
\lim_{j \to +\infty} \beta^j \mathbb{E}_t \frac{\partial V}{\partial K_t}(K_{t+j+1}, \exp(z_{t+j+1})) K_{t+j+1} \geq 0
$$

Remember that we need to impose the transversality condition for the problem to have the same solutions as the recursive problem above.
The conditions characterizing the solution to this problem are:

\[
\begin{align*}
\text{(E.T.)} & \quad \frac{\partial V}{\partial K}(K_t, \exp(z_t)) = \frac{\alpha \exp(z_t)K_t^{\alpha-1}}{C_t} \equiv \frac{R_t}{C_t} \\
\text{(F.O.C)} & \quad \frac{1}{C_t} = \beta \mathbb{E}_t \frac{\partial V}{\partial K}(K_{t+1}, \exp(z_{t+1})) \\
\text{(T.C.)} & \quad \lim_{j \to +\infty} \beta^j \mathbb{E}_t \frac{\partial V}{\partial K}(K_{t+j+1}, \exp(z_{t+j+1}))K_{t+j+1} = 0
\end{align*}
\]

We solve this problem by **guess and verify**:

- Guess that the solution has a certain form - either guess a policy function (in our case, \(C(K_t, \exp(z_t))\)) or guess a value function
- Verify that the guess is consistent with the three conditions above

This method is general, and quite frankly, the only one that will allow you to obtain analytical solutions for DP problems.

### 2.1.1 Policy function guess

Here, we make the guess that:

\[
C(K_t, \exp(z_t)) = \chi \exp(z_t)K_t^\alpha = \chi Y_t
\]

Note that:

\[
R_t = \alpha \frac{Y_t}{K_t}
\]

so that the Euler equation obtained from (E.T.) and (F.O.C.) can be rewritten as:

\[
\begin{align*}
\frac{1}{C_t} & = \beta \mathbb{E}_t \frac{\alpha Y_{t+1}}{K_{t+1}} \frac{1}{C_{t+1}} \\
\frac{1}{C_t} & = \beta \mathbb{E}_t \frac{\alpha Y_{t+1}}{K_{t+1}} \frac{\chi Y_t}{\chi Y_{t+1}} \\
\frac{1}{C_t} & = \beta \mathbb{E}_t \frac{\alpha Y_t}{(1-\chi)Y_t} \\
1 & = \frac{\alpha \beta}{1-\chi} Y_t \\
\chi & = 1 - \alpha \beta
\end{align*}
\]
Therefore, our guess verifies the Euler equation only if $\chi = 1 - \alpha \beta$. It is easy to check that the TVC holds since:

$$\frac{\partial V}{\partial K}(K_t, \exp(z_t))K_t = \frac{\alpha \exp(z_t)K_t^\alpha}{C_t} = \frac{\alpha Y_t}{\chi Y_t} = \frac{\alpha}{1 - \alpha \beta}$$

This policy function is therefore the optimal policy function, for some value function that solves:

$$\frac{\partial V}{\partial K}(K_t, \exp(z_t)) = \frac{\alpha}{(1 - \alpha \beta)K_t}$$

### 2.1.2 Value function guess

We could have gone another way, and guessed that the value function has the form:

$$V(K, \exp(z)) = A \ln(K) + Bz + C$$

Note that this value function form satisfies the transversality condition by construction. Plugging this guess into the E.T. combined with the F.O.C., we get that:

$$A = \frac{\alpha}{1 - \alpha \beta}$$

as in our previous computations. The Euler equation also implies that the policy function for capital is:

$$K_{t+1} = \alpha \beta \exp(z_t)K_t$$

Finally, to solve for $B$ and $C$, one needs to plug back the guess for the value function into the Bellman equation, where the RHS is evaluated at the optimal policy. This leads to:

$$A \ln(K_t) + Bz_t + C = \ln((1 - \alpha \beta) \exp(z_t)K_t^\alpha) + \beta A \ln(\alpha \beta \exp(z)K_t^\alpha) + \beta B \mathbb{E}_t z_{t+1} + \beta C$$

Since this functional equality must hold for all $K_t, z_t$, we can identify coefficients on the right and left hand side. Under the additional assumption that $\mathbb{E}_t z_{t+1} = 0$, we get that:

$$B = \frac{1}{1 - \alpha \beta}$$

$$C = \frac{1}{1 - \beta} \left[ \ln(1 - \alpha \beta) + \frac{\alpha \beta}{1 - \alpha \beta} \log(\alpha \beta) \right]$$
2.2 Precautionary savings in closed form

2.2.1 A simple model

Consider the simple model, directly expressed in Bellman form:

\[ V(W_t) = \max_{C_t} \left( \frac{-\alpha C_t}{\alpha} + \beta E_t V(W_{t+1}) \right) \]

s.t. \[
W_{t+1} = R(W_t - C_t) + Y_{t+1}
\]
\[
W_{t+1} \geq C_{t+1} \geq 0
\]

Note that this implies a borrowing constraint of the form: \( A_{t+1} = W_{t+1} - Y_{t+1} \geq 0 \). Also, note that marginal utility is convex, so that we expect precautionary savings behavior to obtain. Assume \( \beta R = 1 \), and the process for income is:

\[ Y_t \sim N(0, \sigma^2) \]

(Defining a natural debt limit in this case is awkward, since realizations of income may be arbitrarily small. To avoid this type of problem, we could assume that the consumer starts from some sufficiently large level of wealth, and we could put a lower bound on income shocks, that is, draw them from a truncated normal distribution). Following the same steps as previously, the solution to this problem is characterized by:

\[ \frac{\partial V}{\partial W} = \beta R E_t \frac{\partial V}{\partial W}(R(W_t - C_t) + Y_{t+1}) \]

\[ \exp(-\alpha C_t) = \beta R E_t \frac{\partial V}{\partial W}(R(W_t - C_t) + Y_{t+1}) \]

Thus, the Euler equation is:

\[ \exp(-\alpha C_t) = E_t \exp(-\alpha C_{t+1}) \]

As previously, we solve the problem with a guess, this time for the policy function:

\[ C_t = AW_t + B \]

Note that:

\[ C_{t+1} = A(1 - A)RW_t - ABR + AY_{t+1} + B \]

So that:

\[ E_t \exp(-\alpha C_{t+1}) = \exp(-\alpha A(1 - A)RW_t + \alpha ABR - \alpha B) \exp\left(\frac{1}{2} \alpha^2 A^2 \sigma^2\right) \]
Identifying coefficients from the Euler equation, we get that:

\[ A = \frac{R - 1}{R} \]

\[ B = -\frac{1}{2} \frac{R - 1}{R} \alpha \sigma^2 \]

Thus,

\[ C_{t+1} - C_t = \frac{R - 1}{R} Y_{t+1} + \frac{1}{2} \frac{(R - 1)^2}{R^2} \alpha \sigma^2 \]

This equation has lots of nice interpretations. Changes in consumption reflect, first, changes to the present discounted value of future income. If \( R > 1 \), the consumer can increase the present discounted value of income by saving a fraction \( \frac{1}{R} \) of the temporary shock \( Y_t \), and consuming the rest of it. Doing this, he smoothes out the effect of the temporary shock on consumption. The second term reflects precautionary savings. As the variance of income increases, savings increase; this is reflected in the fact that:

\[ S_t = W_t - C_t = \frac{1}{R} W_t + \frac{1}{2} \frac{R - 1}{R} \alpha \sigma^2 \]

where the second part of this expression is the amount of precautionary saving. But since, on average, the realization of the transitory income shock is 0, the additional savings increases expected consumption growth:

\[ \mathbb{E}_t(C_{t+1} - C_t) = \frac{1}{2} \frac{(R - 1)^2}{R^2} \alpha \sigma^2 \]

Note that the amount of precautionary savings increase with absolute risk aversion, \( \alpha \).

A deeper exploration of this model is in Caballero (1996).