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1 Equivalence of the social planner problem solution and the decentralized equilibrium solution in the baseline RBC model

1.1 Two sets of equilibrium conditions

From class, the social planner allocation (SPA) is a set of 3 stochastic processes:

\{C_t, h_t, K_{t+1}\}

that satisfy the following 4 conditions:

\[
\frac{-u_2(C_t, h_t)}{u_1(C_t, h_t)} = z_t A_t F_2(K_t, z_t h_t) \quad (1)
\]

\[
u_1(C_t, h_t) = \beta \mathbb{E}_t [(A_{t+1} F_1(K_{t+1}, z_{t+1} h_{t+1}) + 1 - \delta) u_1(c_{t+1}, h_{t+1})] \quad (2)
\]

\[
A_t F(K_t, z_t h_t) = C_t + K_{t+1} - (1 - \delta) K_t \quad (3)
\]

\[
\lim_{j \to +\infty} \mathbb{E}_t \beta^{j} u_1(C_{t+j}, h_{t+j}) K_{t+j+1} = 0 \quad (4)
\]

given exogenous processes for \(A_t\) and \(z_t\) and an initial condition \(K_0\).

The decentralized equilibrium (DE) is a set of 8 stochastic processes:

\{C_t, h_t, K_{t+1}, u_t, W_t, r_{t,t+1}, d_{t+1}, \lambda_t\}

that satisfy the following 9 conditions:

\[
U_1(C_t, \tilde{h}_t) = \lambda_t \quad (1')
\]

\[
U_2(C_t, \tilde{h}_t) = -\lambda_t W_t \quad (2')
\]

\[
W_t = A_t F_1(K_t, h_t) \quad (3')
\]

\[
u_t = A_t F_2(K_t, h_t) \quad (4')
\]

\[
K_{t+1} = \mathbb{E}_t r_{t,t+1} d_{t+1} \quad (5')
\]

\[
r_{t,t+1} = \beta \frac{\lambda_{t+1}}{\lambda_t} \text{ a.s.} \quad (6')
\]

\[
K_{t+1}(1 - \delta + u_{t+1}) = d_{t+1} \quad (7')
\]

\[
C_t + \mathbb{E}_t r_{t,t+1} d + t + 1 = W_t h_t + d_t \quad (8')
\]

\[
\lim_{x \to +\infty} \mathbb{E}_t r_{t,t+j} d_{t+j} = 0 \quad (9')
\]
1.2 There is a decentralized equilibrium supporting each planner allocation

We need to show that given:

\[ \{C_t, h_t, K_{t+1}\} \]

satisfying (1)-(4), we can construct

\[ \{w_t, u_t, r_{t,t+1}, d_{t+1}, \lambda_t\} \]

such that:

\[ \{C_t, h_t, K_{t+1}, u_t, W_t, r_{t,t+1}, d_{t+1}, \lambda_t\} \]

satisfies (1')-(9').

**Step a):** Define:

\[ \lambda_t \equiv u_1(c_t, h_t) \]

\[ w_t \equiv A_t z_t F_2(K_t, z_t h_t) \]

From (1) and (2), (1')-(3') are verified.

**Step b):** Define:

\[ u_t \equiv A_t F_1(K_t, z_t h_t) \]

\[ d_t \equiv (1 - \delta + u_t)K_t \]

By homogeneity of degree 1 of the production function,

\[ A_t F(K_t, z_t h_t) = u_t K_t + w_t h_t \]

so that, from (3):

\[ u_t K_t + w_t h_t = C_t + K_{t+1} - (1 - \delta)K_{t+1} \]

or:

\[ w_t h_t + d_t = C_t + K_{t+1} \]
**Step c):** Define:

\[ r_{t,t+1} = \beta \frac{u_1(c_{t+1}, h_{t+1})}{u_1(c_t, h_t)} \]

which is (6'). Then, by (2),

\[ 1 = \mathbb{E}_t r_{t,t+1}(1 - \delta + u_{t+1}) \]
\[ = \mathbb{E}_t r_{t,t+1} \frac{d_{t+1}}{K_{t+1}} \]

As \( K_{t+1} \) is determined in period \( t \),

\[ K_{t+1} = \mathbb{E}_t r_{t,t+1}d_{t+1} \]

so that (5') and therefore (8') are satisfied.

**Step d):** Note that:

\[ \beta^{j-1}u_1(t + j - 1)K_{t+j} = u_1(t) \left( \prod_{k=1}^{j-1} \beta \frac{u_1(t + k)}{u_1(t + k - 1)} \right) K_{t+j} \]
\[ = u_1(t) \left( \prod_{k=1}^{j-1} r_{t+k-1,t+k} \right) K_{t+j} \]
\[ = u_1(t)r_{t,t+j-1}K_{t+j} \]

Furthermore, by the law of iterated expectations and (5'),

\[ \mathbb{E}_t u_1(t)r_{t,t+j-1}K_{t+j} = u_1(t)\mathbb{E}_t r_{t,t+j-1}\mathbb{E}_{t+j-1} \mathbb{E}_{t+j-1} r_{t+j-1,t+j}d_{t+j} \]
\[ = u_1(t)\mathbb{E}_t r_{t,t+j-1}d_{t+j} \]
\[ = u_1(t)\mathbb{E}_t r_{t,t+j}d_{t+j} \]

Since by (4),

\[ \lim_{j \to +\infty} \mathbb{E}_t u_1(t + j - 1)K_{t+j} = 0 \]

we have:

\[ u_1(t) \lim_{j \to +\infty} \mathbb{E}_t u_1(t + j - 1)K_{t+j} = \lim_{j \to +\infty} u_1(t)\mathbb{E}_t u_1(t + j - 1)K_{t+j} = \lim_{j \to +\infty} \mathbb{E}_t r_{t,t+j}d_{t+j} = 0 \]

which is (9').
1.3 Each decentralized equilibrium allocation is a planner allocation

We need to show that given a decentralized equilibrium

\[ \{ C_t, h_t, K_{t+1}, u_t, W_t, r_{t,t+1}, d_{t+1}, \lambda_t \} \]

the allocation

\[ \{ C_t, h_t, K_{t+1} \} \]

takes place

satisfies (1)-(4).

**Step a)**: Note that:

\[
\mathbb{E}_t r_{t,t+j+1}d_{t+j+1} = \mathbb{E}_t \mathbb{E}_{t+j} r_{t,t+j+1}d_{t+j+1} = \mathbb{E}_t r_{t,t+j} \mathbb{E}_{t+j} r_{t+j,t+j+1}d_{t+j+1} = \mathbb{E}_t r_{t,t+j}K_{t+j+1}
\]

by the law of iterated expectations and (5'). Using (6'),

\[
\mathbb{E}_t r_{t,t+j+1}d_{t+j+1} = \mathbb{E}_t r_{t,t+j}K_{t+j} = \mathbb{E}_t \beta^j \frac{u_1(t+j)}{u_1(t)} K_{t+j+1}
\]

Multiplying by (9') by \( u_1(t) \), this implies

\[
\lim_{j \to +\infty} \mathbb{E}_t \beta^j u_{1,t+j}K_{t+j+1} = 0
\]

which is (4').

**Step b)**: (1'),(2'),(3') imply (1).

**Step c)**: (3'),(4'), (5'), (7'), (8'), and homogeneity of degree 1 of \( F \) implies (3).

**Step d)**: (1'),(4'),(6') and (7') imply (4).
2 Review of monopolistic pricing and Dixit-Stiglitz aggregators

Consider the following static production problem. There is one final good and a large number of intermediate goods, indexed by $i \in [0, 1]$. Aggregate demand for the final good, $c$, is given. The price of the final good is denoted by $p$. $c_i$ is the quantity of each intermediate variety produced.

Production of the final good is realized by aggregating varieties according to:

$$c = \left( \int_0^1 c_i^\gamma di \right)^{\frac{1}{\gamma}}$$

where $\gamma < 1$. The final good production sector is perfectly competitive.

Each intermediate good producer is a monopolist and sells his variety at price $p_i$.

2.1 Final good producer

Final good producers take $p$ and $c$ as given. Their profit maximization problem is equivalent to the cost minimization problem:

$$\min_{c_i} \int_0^1 p_i c_i di \quad \text{s.t.} \quad (\mu) \quad \left( \int_0^1 c_i^\gamma di \right)^{\frac{1}{\gamma}} \geq c$$

$\mu$ is the marginal cost of the final firm. The solution to this problem is:

$$\mu = \left( \int_0^1 \frac{c_i}{p_i^\frac{1}{\gamma}} di \right)^{\frac{\gamma-1}{\gamma}} c_i = \left( \frac{p_i}{\mu} \right)^{\frac{1}{1-\gamma}} c$$

Furthermore, perfect competition implies that:

$$\mu = p$$

In this setup, the price elasticity of demand for each intermediate good is:

$$\epsilon = \frac{1}{1 - \gamma}$$
It is equal to the elasticity of substitution across varieties, since:

\[ \frac{c_i}{c_j} = \left( \frac{p_i}{p_j} \right)^{-\frac{1}{\gamma}} \]

### 2.2 Intermediate monopolist

The intermediate monopolist for variety \( i \) maximizes profits given the demand schedule for his variety:

\[ c_i(p_i) = \left( \frac{p_i}{\mu} \right)^{-\frac{1}{\gamma}} c \]

and his production possibilities, summarized by \( F(x_{i,1}, ..., x_{i,n}) \). Each input type \( x_{i,j} \) has price \( q_j \). His problem can be written as:

\[
\max_{x_{1, \ldots, x_{i,n}, p_i}} \quad p_i c_i(p_i) - \sum_{j=1}^{n} q_j x_j \\
\text{s.t.} \quad F(x_{i,1}, ..., x_{i,n}) \leq c_i(p_i) \quad (\mu_i)
\]

\( \mu_i \) is the marginal cost of the intermediate monopolist. With monopoly pricing, equalizing marginal revenue to marginal cost implies setting the price above the marginal cost, because marginal revenue is given by:

\[ m_i = p_i + c_i \frac{dp_i}{dc_i} \]

and the demand schedule is decreasing (so that \( m_i < p_i \)). With our constant elasticity demand schedule, the monopoly pricing condition is:

\[ p_i = \frac{1}{\gamma} \mu_i \]

\( \theta \equiv \frac{1}{\gamma} \) is called the "markup" of price over marginal cost. Note that, as in any standard profit maximization problem, the monopolist producer still equates marginal costs across factor inputs:

\[ \forall j, \quad \mu_i = \frac{q_j}{F_j} \]

Assuming that the production possibilities are homogeneous of degree 1, profits are given by:

\[ \Pi_i = \left( \frac{1}{\gamma} - 1 \right) p_i c_i > 0 \]