Contents

1 What is log-linearization? Some basic results

Log-linearization is a first-order Taylor expansion, expressed in percentage terms rather than in levels differences.

In Economics, since units are not always well defined or consistent, we prefer to think in terms of percentage deviations from reference values. This reference value will very often be a steady-state of a model that we are studying.

To fix ideas, consider the following relationship between three level variables (x, y, z):

$$z = f(x, y)$$

Taking a first order Taylor expansion around some point (x_0, y_0, z_0), where $z_0 = f(x_0, y_0)$, expressed in levels, would consist in writing:

$$z - z_0 = f_1(x_0, y_0)(x - x_0) + f_2(x_0, y_0)(y - y_0) + o(\|x - x_0\|, \|y - y_0\|)$$

This is a first-order approximation of the level change in z around z_0, as a function of the level changes $x - x_0$, $y - y_0$.

Instead, we want to look at the *percentage* change in z, relative to z_0, as x and y change around (x_0, y_0). To do this, define:

$$\hat{z} = \log\left(\frac{z}{z_0}\right) \approx \frac{z - z_0}{z_0}$$

$$\hat{x} = \log\left(\frac{x}{x_0}\right)$$

$$\hat{y} = \log\left(\frac{y}{y_0}\right)$$

Taking a first-order Taylor expansion of:

$$f(x, y) = f(x_0 \exp(\hat{x}), y_0 \exp(\hat{y}))$$

around $(\hat{x}, \hat{y}) = (0, 0)$, we get:

$$f(x, y) - f(x_0, y_0) = x_0 f_1(x_0, y_0)\hat{x} + y_0 f_2(x_0, y_0) + o(\|\hat{x}\|, \|\hat{y}\|)$$

Dividing through by $z_0 = f(x_0, y_0)$, we get:

$$\hat{z} \approx= \frac{z - z_0}{z_0} = \frac{x_0 f_1(x_0, y_0)}{f(x_0, y_0)}\hat{x} + \frac{y_0 f_2(x_0, y_0)}{f(x_0, y_0)} + o(\|\hat{x}\|, \|\hat{y}\|)$$

We have obtained our log-linear approximation. The coefficients in front of $\hat{x}$ and $\hat{y}$ are the elasticities of f with respect to x and y, evaluated at (x_0, y_0).

We can generalize this result to many variables.

Result 1 (Log-linearization)**.** *The first-order log-linear approximation of the relationship:*

$$z = f(x_1, ..., x_n)$$

around the point $(x_1^0, ..., x_n^0)$, $z_0 = f(x_1^0, ..., x_n^0)$ *is:*

$$\hat{z} = \sum_i \epsilon_{f,i}\hat{x}_i$$

where:

$$\hat{x}_i = \log\left(\frac{x_i}{x_i^0}\right)$$

$$\hat{z} = \log\left(\frac{z}{z_0}\right)$$

and where:

$$\epsilon_{f,i} = \frac{f_i(x_1^0, ..., x_n^0)x_i^0}{f(x_1^0, ..., x_n^0)} = \frac{f_i(x_1^0, ..., x_n^0)x_i^0}{z_0}$$

is the elasticity of f with respect to its $i - th$ argument, evaluated at $(x_1^0, ..., x_n^0)$.

This corrolary may serve as a shortcut in many computations.

Result 2 (Some useful relations)**.**

- *If*

$$z = y^\alpha x^\beta$$

  then

$$\hat{z} = \alpha\hat{y} + \beta\hat{x}$$

- *If*

$$z = \sum_i \alpha_i x_i$$

then

$$\hat{z} = \sum_i \frac{\alpha_i x_i^0}{\sum_i \alpha_i x_i^0} \hat{x}_i = \sum_i \frac{\alpha_i x_i^0}{z_0} \hat{x}_i$$

2 Application: the baseline RBC model

The REE of the baseline RBC model can be summarized as the set of following relations:

$$u_c(c_t, h_t) = \lambda_t \tag{MU = SV of income}$$
$$u_h(c_t, h_t) = -\lambda_t w_t \tag{MDL = SV of income $\times$ Wage}$$
$$w_t = a_t k_t^{\alpha} h_t^{1-\alpha} \tag{MPL = Wage}$$
$$c_t + k_{t+1} - (1-\delta)k_t = a_t k_t^{\alpha} h_t^{1-\alpha} \tag{Ressource constraint}$$
$$\lambda_t = \beta \mathbb{E}_t \lambda_{t+1} \left(1 - \delta + \alpha a_{t+1} k_{t+1}^{\alpha-1} h_{t+1}^{1-\alpha}\right) \tag{Euler equation}$$

Using our previous results, it is straightforward to log-linearize these relations around the deterministic steady-state of the model. In what follows, any elasticity is evaluated at the deterministic steady-state of the model, and any variable without a time subscript is a steady-state value.

The log-linearization of the first-order condition with respect to consumption follows from the main result directly:

$$\hat{\lambda}_t = \frac{u_{cc} c}{u_c} \hat{c}_t + \frac{u_{ch} h}{u_c} \hat{h}_t$$

The right-hand side of the FOC with respect to hours can be log-linearized using the first part of the corrolary, to obtain:

$$\hat{\lambda}_t + \hat{w}_t$$

while the log-linear form of the left hand side is obtained from the main result. The log-linear form of the FOC with respect to hours is thus:

$$\hat{\lambda}_t + \hat{w}_t = \frac{u_{ch} c}{u_h} \hat{c}_t + \frac{u_{hh} h}{u_h} \hat{h}_t$$

The log-linear form of the labour demand schedule follows directly from the first part of the corrolary:

$$\hat{w}_t = \hat{a}_t + \alpha(\hat{k}_t - \hat{h}_t)$$

The right hand side of the ressource constraint is log-linearized using the first part of the corrolary as well. For the left-hand side, we use the second additional part of the corrolary. The log-linear form of the ressource constraint is then:

$$\frac{c}{y}\hat{c}_t + \frac{k}{y}(\hat{k}_t - (1-\delta)\hat{k}_{t+1}) = \hat{a}_t + \alpha\hat{k}_t + (1-\alpha)\hat{h}_t$$

where $y = c + \delta k$.

The Euler equation is slightly more difficult to log-linearize. Rewrite is as:

$$\frac{\lambda_t}{\beta} = \mathbb{E}_t \lambda_{t+1}\left(1 - \delta + \alpha a_{t+1} k_{t+1}^{\alpha-1} h_{t+1}^{1-\alpha}\right)$$

The left hand side is log-linearized as:

$$\hat{\lambda}_t$$

Consider log-linearizing:

$$\lambda_{t+1}\left(1 - \delta + \alpha a_{t+1} k_{t+1}^{\alpha-1} h_{t+1}^{1-\alpha}\right)$$

We can use either the first result, or a combination of the additional results, to get the log-linear form:

$$\frac{1}{\Gamma}\left[\lambda\left(1-\delta+\alpha k^{\alpha-1}h^{1-\alpha}\right)\hat{\lambda}_{t+1} + \lambda\alpha k^{\alpha-1}h^{1-\alpha}\left(\hat{a}_{t+1} + (1-\alpha)(\hat{h}_{t+1} - \hat{k}_{t+1})\right)\right]$$

where:

$$\Gamma = \lambda(1-\delta+\alpha A k^{\alpha-1}h^{1-\alpha}) = \lambda/\beta$$

Simplifying and replacing into the expectation, the log-linear form of the Euler equation is:

$$\hat{\lambda}_t = \mathbb{E}_t\hat{\lambda}_{t+1} + \beta\alpha k^{\alpha-1}h^{1-\alpha}\left(\mathbb{E}_t\hat{a}_{t+1} + (1-\alpha)\left(\mathbb{E}_t\hat{h}_{t+1} - \hat{k}_{t+1}\right)\right)$$

In log-linear form, the REE of the RBC model can therefore be written as:

$$\hat{\lambda}_t = \frac{u_{cc}c}{u_c}\hat{c}_t + \frac{u_{ch}h}{u_c}\hat{h}_t \qquad \text{(MU = SV of income)}$$

$$\hat{\lambda}_t + \hat{w}_t = \frac{u_{ch}c}{u_h}\hat{c}_t + \frac{u_{hh}h}{u_h}\hat{h}_t \qquad \text{(MDL = SV of income} \times \text{Wage)}$$

$$\hat{w}_t = \hat{a}_t + \alpha(\hat{k}_t - \hat{h}_t) \qquad \text{(MPL = Wage)}$$

$$\frac{c}{y}\hat{c}_t + \frac{k}{y}(\hat{k}_t - (1-\delta)\hat{k}_{t+1}) = \hat{a}_t + \alpha\hat{k}_t + (1-\alpha)\hat{h}_t \qquad \text{(Ressource constraint)}$$

$$\hat{\lambda}_t = \mathbb{E}_t\hat{\lambda}_{t+1} + \beta\alpha k^{\alpha-1}h^{1-\alpha}\left(\mathbb{E}_t\hat{a}_{t+1} + (1-\alpha)\left(\mathbb{E}_t\hat{h}_{t+1} - \hat{k}_{t+1}\right)\right) \qquad \text{(Euler equation)}$$

Note that the block of the first two FOC can be inverted to express $(\hat{c}_t, \hat{h}_t)$ as a function of $(\hat{w}_t, \hat{\lambda}_t)$. This is the point of the next homework.