Physics G6037 | Professor Christ
due 09/22/2017

Problem Set #2

Be careful not to miss the problems on page 2.

3. Consider a 3-dimensional quantum mechanical Hilbert space and the operator O represented by the matrix:

$$O = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

   (a) Find the eigenvectors and eigenvalues of O.

   (b) What is the probability of finding each of these eigenvalues if the quantity O is measured for the state vector:

$$|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}?$$

4. If $E_F(\lambda)$ is a resolution of the identity for the self-adjoint operator F, $F = \int \lambda \, dE_F(\lambda)$.

   (a) Show that $\langle\psi|E(\lambda)|\psi\rangle$ is a non-decreasing, non-negative function of λ.

   (b) Find the resolution of the identity, $E_{F^2}(\lambda)$ for the operator F^2 in terms of $E_F(\lambda)$. [Although not very difficult, this part does require careful thought.]

5. Consider two self-adjoint operators A and B. Assume that spectrum of both A and B is discrete with a corresponding complete sets of eigenvectors:

$$\begin{aligned} A|n\rangle_A &= a_n|n\rangle_A \\ B|n\rangle_B &= b_n|n\rangle_B. \end{aligned}$$

   If A and B have a vanishing commutator, $[A, B] = 0$, show that A and B can be simultaneously diagonalized, *i.e.* there exists a common set of eigenvectors for both operators for each of the following situations of increasing difficulty:

   (a) The spectrum of A and B is non-degenerate, *e.g.* for each eigenvalue of A there exists only a single eigenvector of A with that eigenvalue.

   (b) All degeneracies are finite dimensional, *e.g.* for each eigenvalue of A there exists only a finite number of independent eigenvectors of A with that eigenvalue.

   (c) Even infinite dimensional subspaces of eigenvectors of A or B with the same eigenvalue are allowed. [This is an awkwardly phrased problem which appears to be difficult.]

6. Consider a hermitian operator O acting on a complex vector space of finite dimension N. Consider a "sphere" of states $|\psi\rangle$ with unit norm, $||\ |\psi\rangle\ ||^2 = 1$. Make the reasonable assumption that there should exist a vector $|\psi_0\rangle$ which lies on this sphere for which the expression,

$$\Big(|\psi_0\rangle, O|\psi_0\rangle\Big) = \langle\psi_0|O|\psi_0\rangle \tag{1}$$

takes a value less than or equal to the value of $\langle\psi|O|\psi\rangle$ found for any state $|\psi\rangle$ on this sphere. (Prove the truth of this assumption if you wish.) Consider a second, unit vector $|\delta\psi\rangle$ which lies in the $N-1$ dimensional space of vectors perpendicular to $|\psi_0\rangle$:

$$\Big(|\delta\psi\rangle, |\psi_0\rangle\Big) = \langle\delta\psi|\psi_0\rangle = 0. \tag{2}$$

(a) Define the vector

$$|\psi_\epsilon\rangle = \frac{1}{\sqrt{1+|\epsilon|^2}}\Big(|\psi_0\rangle + \epsilon|\delta\psi\rangle\Big) \tag{3}$$

where ϵ is a complex number. Show that the state $|\psi_\epsilon\rangle$ also has unit length.

(b) Since $|\psi_0\rangle$ is a state of unit length which minimizes the expectation value of O among those states of unit length we can conclude that

$$\Big(|\psi_\epsilon\rangle, O|\psi_\epsilon\rangle\Big) \geq \Big(|\psi_0\rangle, O|\psi_0\rangle\Big). \tag{4}$$

Show that if evaluated to first order in the small complex number ϵ this inequality implies that for all states $|\delta\psi\rangle$

$$\Big(|\delta\psi\rangle, O|\psi_0\rangle\Big) = 0. \tag{5}$$

(c) Explain how this result implies that $O|\psi_0\rangle$ must be proportional to $|\psi_0\rangle$ or

$$O|\psi_0\rangle = \lambda_0|\psi_0\rangle, \tag{6}$$

if the proportionality constant is labeled λ_0, proving O has at least one eigenvector.

(d) Consider the $N-1$ dimensional space of vectors that are orthogonal to $|\psi_0\rangle$ and repeat the above argument to find a second eigenvector $|\psi_1\rangle$ of O.

(e) Repeating the above procedure, deduce that any hermitian O acting on a finite dimensional vector space of dimension N can be diagonalized.

7. Prove the Schwartz inequality:

$$|\langle A|B\rangle|^2 \leq \langle A|A\rangle\langle B|B\rangle.$$

[Hint: you might use a method similar to our proof of the uncertainty relation from the commutator $[x,p] = i\hbar$, examining the norm of the vector $|A\rangle + \alpha|B\rangle$ as a function of the variable α.]