

Problem Set #2
(Revised with 5. simplified and 7. delayed)

3. Consider a 3-dimensional quantum mechanical Hilbert space and the operator O represented by the matrix:

$$O = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

- (a) Find the eigenvectors and eigenvalues of O .
 (b) What is the probability of finding each of these eigenvalues if the quantity O is measured for the state vector:

$$|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}?$$

4. If $E_F(\lambda)$ is a resolution of the identity for the self-adjoint operator F , $F = \int \lambda dE_F(\lambda)$.
 (a) Show that $\langle\psi|E(\lambda)|\psi\rangle$ is a non-decreasing, non-negative function of λ .
 (b) Find the resolution of the identity, $E_{F^2}(\lambda)$ for the operator F^2 in terms of $E_F(\lambda)$. [Although not very difficult, this part does require careful thought.]
5. Consider two self-adjoint operators A and B . Assume that spectrum of both A and B is discrete with a corresponding complete sets of eigenvectors:

$$\begin{aligned} A|n\rangle_A &= a_n|n\rangle_A \\ B|n\rangle_B &= b_n|n\rangle_B. \end{aligned}$$

If A and B have a vanishing commutator, $[A, B] = 0$, show that A and B can be simultaneously diagonalized, *i.e.* there exists a common set of eigenvectors for both operators for each of the following situations of increasing difficulty:

- (a) The spectrum of A and B is non-degenerate, *e.g.* for each eigenvalue of A there exists only a single eigenvector of A with that eigenvalue.
 (b) All degeneracies are finite dimensional, *e.g.* for each eigenvalue of A there exists only a finite number of independent eigenvectors of A with that eigenvalue.

(While not part of this problem set you might try extend this result to the case that infinite dimensional subspaces of eigenvectors of A or B with the same eigenvalue are allowed. This is an awkwardly phrased problem which appears to be difficult.)

6. Consider a hermitian operator O acting on a complex vector space of finite dimension N . Consider a “sphere” of states $|\psi\rangle$ with unit norm, $\| |\psi\rangle \|^2 = 1$. Make the reasonable assumption that there should exist a vector $|\psi_0\rangle$ which lies on this sphere for which the expression,

$$\left(|\psi_0\rangle, O|\psi_0\rangle \right) = \langle \psi_0 | O | \psi_0 \rangle \quad (1)$$

takes a value less than or equal to the value of $\langle \psi | O | \psi \rangle$ found for any state $|\psi\rangle$ on this sphere. (Prove the truth of this assumption if you wish.) Consider a second, unit vector $|\delta\psi\rangle$ which lies in the $N - 1$ dimensional space of vectors perpendicular to $|\psi_0\rangle$:

$$\left(|\delta\psi\rangle, |\psi_0\rangle \right) = \langle \delta\psi | \psi_0 \rangle = 0. \quad (2)$$

- (a) Define the vector

$$|\psi_\epsilon\rangle = \frac{1}{\sqrt{1 + |\epsilon|^2}} \left(|\psi_0\rangle + \epsilon |\delta\psi\rangle \right) \quad (3)$$

where ϵ is a complex number. Show that the state $|\psi_\epsilon\rangle$ also has unit length.

- (b) Since $|\psi_0\rangle$ is a state of unit length which minimizes the expectation value of O among those states of unit length we can conclude that

$$\left(|\psi_\epsilon\rangle, O|\psi_\epsilon\rangle \right) \geq \left(|\psi_0\rangle, O|\psi_0\rangle \right). \quad (4)$$

Show that if evaluated to first order in the small complex number ϵ this inequality implies that for all states $|\delta\psi\rangle$

$$\left(|\delta\psi\rangle, O|\psi_0\rangle \right) = 0. \quad (5)$$

- (c) Explain how this result implies that $O|\psi_0\rangle$ must be proportional to $|\psi_0\rangle$ or

$$O|\psi_0\rangle = \lambda_0 |\psi_0\rangle, \quad (6)$$

if the proportionality constant is labeled λ_0 , proving O has at least one eigenvector.

- (d) Consider the $N - 1$ dimensional space of vectors that are orthogonal to $|\psi_0\rangle$ and repeat the above argument to find a second eigenvector $|\psi_1\rangle$ of O .
- (e) Repeating the above procedure, deduce that any hermitian O acting on a finite dimensional vector space of dimension N can be diagonalized.