Physics G6037 Professor Christ
due 10/26/2018

Problem Set #7

Don't miss the two problems on the second page.

22. Consider a quantum mechanical system with total angular momentum $j = \frac{1}{2}$ (for example a proton at rest with intrinsic spin $j = \frac{1}{2}$). Any state $|\psi\rangle$ of the system can be written as a superposition of the two states $|j = \frac{1}{2}, m = \pm\frac{1}{2}\rangle$:

$$|\psi\rangle = a_+|\frac{1}{2}, \frac{1}{2}\rangle + a_-|\frac{1}{2}, -\frac{1}{2}\rangle.$$

where a_+ and a_- are complex numbers.

(a) If the system is known to be in the state with $J_3 = +\frac{\hbar}{2}$ what is the probability that a measurement of $J_3\cos(\theta) + J_2\sin(\theta)$ will yield the value $+\frac{\hbar}{2}$?

(b) If $\hat{n}$ is a unit vector pointing in a direction specified by the polar coordinates θ and ϕ find values for the two coefficients a_+ and a_- which determine a state $|\psi(\theta, \phi)\rangle$ which obeys

$$\langle\psi(\theta, \phi)|\vec{J}|\psi(\theta, \phi)\rangle = \frac{\hbar}{2}\hat{n}.$$

(c) If the time development of this spin-$\frac{1}{2}$ system is determined by the Hamiltonian

$$H = -\frac{ge}{2mc}\vec{J}\cdot\vec{B}$$

(for example our proton sits in a magnetic field $\vec{B}$), and it is known that at $t = 0$, $J_3 = +\frac{\hbar}{2}$, find the resulting state $|\psi(t)\rangle$ at a later time t. (Note, $\vec{B}$ should be assumed to point in a general direction.)

(d) For the case that $\vec{B} = |\vec{B}|\hat{z}$ and the initial state is the state $|\psi(\theta, \phi)\rangle$ found in part (b), find the resulting state at a later time t and show that the expectation value $\langle\psi(t)|\vec{J}|\psi(t)\rangle$ has the same time dependence as a classical magnetic dipole with angular momentum $\vec{J}$ and magnetic moment $\vec{\mu} = \frac{ge}{2mc}\vec{J}$ precessing in the magnetic field $\vec{B}$.

23. Let $|+\frac{1}{2}\rangle$ and $|-\frac{1}{2}\rangle$ be two eigenstates of J_3 for a spin-$\frac{1}{2}$ particle with eigenvalues $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$.

(a) Show that for any normalized spin-$\frac{1}{2}$ state $|\psi\rangle$, $\langle\psi|\vec{J}|\psi\rangle\frac{2}{\hbar}$ is a unit vector.

(b) For $j > \frac{1}{2}$, is $\langle\psi|\vec{J}|\psi\rangle\frac{1}{j\hbar}$ a unit vector? Give a proof or a counter example.

24. Consider functions $F(x, y, z)$ depending on the three spatial coordinates x, y and z which have the form:

$$F(x, y, z) = \sum_{n_x=0}^{N} \sum_{n_y=0}^{N} \sum_{n_z=0}^{N} c_{n_x n_y n_z} x^{n_x} y^{n_y} z^{n_z}$$

where the coefficients $c_{n_x n_y n_z}$ are complex constants, non-vanishing only when the sum of the three integers $n_x + n_y + n_z = N$, *i.e.* the function F is a polynomial of order N.

(a) Show that if a polynomial with a fixed value of N is viewed as a quantum mechanical wave function and acted on by a rotation operator, it will transform into a second polynomial which is also of order N. Conclude that the vector space of all such polynomials with a fixed value of N will form a representation of the rotation group.

(b) If N is even, find that function of the above form with $L^2 = \hbar^2 l(\ell + 1) = 0$.

(c) If N is odd, find those functions with $\ell = 1$, $m = -1, 0, +1$. (Hint: first consider the case $N = 1$ and compare with the functions $rY_{1,m}(\theta, \phi)$.)

(d) Find the function with $\ell = N$ and $m = +N$.

25. Consider the 3-dimensional harmonic oscillator with

$$H = \frac{\vec{p}^{\,2}}{2m} + \frac{1}{2} m\omega^2 \vec{r}^{\,2}.$$

(a) Find the allowed energies and corresponding multiplicities for this Hamiltonian.

(b) Observe that the eigenstates of H can be chosen as eigenstates of L^2 and L_z as well. What values of ℓ and m, with what multiplicities are possible for states with a given energy eigenvalue E? (Hint: this can be answered by counting the number of states with a given energy and a given value of ℓ, referring to the solution to problem 24 above and discussing the properties of the radial Schrödinger equation implied by H.)

(c) Determine the form of the radial wave function for those states with $E = \hbar\omega(N + \frac{3}{2})$ and $\ell = N$.