

Problem Set #7

Don't miss the last problem on the second page.

22. Find the three matrices $(J_i)_{mm'} = \langle j, m | J_i | j, m' \rangle$, $1 \leq i \leq 3$, explicitly for the case $j = 1$. Verify the commutation relation: $[J_1, J_2] = i\hbar J_3$.
23. Consider a quantum mechanical system with total angular momentum $j = \frac{1}{2}$ (for example a proton at rest with intrinsic spin $j = \frac{1}{2}$). Any state $|\psi\rangle$ of the system can be written as a superposition of the two states $|j = \frac{1}{2}, m = \pm \frac{1}{2}\rangle$:

$$|\psi\rangle = a_+ |\frac{1}{2}, \frac{1}{2}\rangle + a_- |\frac{1}{2}, -\frac{1}{2}\rangle.$$

where a_+ and a_- are complex numbers.

- (a) If the system is known to be in the state with $J_3 = +\frac{\hbar}{2}$ what is the probability that a measurement of $J_3 \cos(\theta) + J_2 \sin(\theta)$ will yield the value $+\frac{\hbar}{2}$?
- (b) If $\hat{n}$ is a unit vector pointing in a direction specified by the polar coordinates θ and ϕ find values for the two coefficients a_+ and a_- which determine a state $|\psi(\theta, \phi)\rangle$ which obeys

$$\langle \psi(\theta, \phi) | \vec{J} | \psi(\theta, \phi) \rangle = \frac{\hbar}{2} \hat{n}.$$

- (c) If the time development of this spin- $\frac{1}{2}$ system is determined by the Hamiltonian

$$H = -\frac{ge}{2mc} \vec{J} \cdot \vec{B}$$

(for example our proton sits in a magnetic field $\vec{B}$), and it is known that at $t = 0$, $J_3 = +\frac{\hbar}{2}$, find the resulting state $|\psi(t)\rangle$ at a later time t . (Note, $\vec{B}$ should be assumed to point in a general direction.)

- (d) For the case that $\vec{B} = |\vec{B}| \hat{z}$ and the initial state is the state $|\psi(\theta, \phi)\rangle$ found in part (b), find the resulting state at a later time t and show that the expectation value $\langle \psi(t) | \vec{J} | \psi(t) \rangle$ has the same time dependence as a classical magnetic dipole with angular momentum $\vec{J}$ and magnetic moment $\vec{\mu} = \frac{ge}{2mc} \vec{J}$ precessing in the magnetic field $\vec{B}$.
24. Let $|+\frac{1}{2}\rangle$ and $|-\frac{1}{2}\rangle$ be two eigenstates of J_3 for a spin- $\frac{1}{2}$ particle with eigenvalues $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$.
- (a) Show that for any normalized spin- $\frac{1}{2}$ state $|\psi\rangle$, $\langle \psi | \vec{J} | \psi \rangle \frac{2}{\hbar}$ is a unit vector.
- (b) For $j > \frac{1}{2}$, is $\langle \psi | \vec{J} | \psi \rangle \frac{1}{j\hbar}$ a unit vector? Give a proof or a counter example.

25. Consider functions $F(x, y, z)$ depending on the three spatial coordinates x , y and z which have the form:

$$F(x, y, z) = \sum_{n_x=0}^N \sum_{n_y=0}^N \sum_{n_z=0}^N c_{n_x n_y n_z} x^{n_x} y^{n_y} z^{n_z}$$

where the coefficients $c_{n_x n_y n_z}$ are complex constants, non-vanishing only when the sum of the three integers $n_x + n_y + n_z = N$, *i.e.* the function F is a polynomial of order N .

- (a) Show that if a polynomial with a fixed value of N is viewed as a quantum mechanical wave function and acted on by a rotation operator, it will transform into a second polynomial which is also of order N . Conclude that the vector space of all such polynomials with a fixed value of N will form a representation of the rotation group.
- (b) If N is even, find that function of the above form with $L^2 = \hbar^2 l(l+1) = 0$.
- (c) If N is odd, find those functions with $\ell = 1$, $m = -1, 0, +1$. (Hint: first consider the case $N = 1$ and examine with the functions $rY_{1,m}(\theta, \phi)$.)
- (d) Find the function with $\ell = N$ and $m = +N$.
- (e) Using the examples above, construct an inductive argument showing that the vector space of polynomials of order N is the same as the vector space constructed from the set of functions $r^{N-\ell}Y_{\ell,m}$, for all ℓ obeying $0 \leq \ell \leq N$ where ℓ is even if N is even and ℓ is odd if N is odd. [Hint: It may be useful to show that the dimension of these two vector spaces increase by the same amount when N is increased by 2.]