Physics GR6037 | Professor Christ
due 09/13/2024

Problem Set #1

Don't miss the second problem on the next page.

1. Consider a two-dimensional problem in which a beam of particles moves to the right in the $+x$ direction with momentum p. The particles in the beam scatter from an array of $N+1$ atoms arranged along the y axis with coordinates $(x=0, y=jd)$ where j is an integer in the range $-N/2 \le j \le N/2$, d is the separation of the atoms in the vertical direction and N is even. Assume that each of the scattered particles can be represented by a wave which is the superposition of the $N+1$ circular waves:

$$\psi(x,y) = A \sum_{j=-N/2}^{N/2} e^{i\sqrt{x^2+(y-jd)^2}\,\frac{p}{\hbar}}.$$

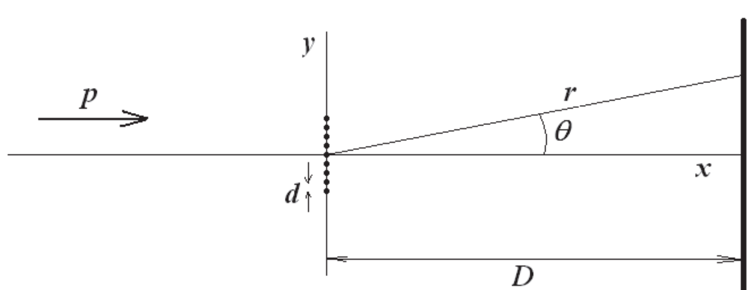
   (a) In the limit that the distance $r = \sqrt{x^2+y^2}$ is large but $\sin(\theta) = y/r$ is small and fixed, expand the exponents in the above expression for $\psi(x,y)$, retaining the term proportional to r and the term which is a constant as r becomes large.

   (b) Using this approximation, perform the sum over j and determine the resulting interference pattern that would would be seen on a screen a large distance D from the atoms.

   (c) Finally replace the array of atoms by a vertical, attenuating screen placed along the y axis. Assume that the incident wave passes through this screen with probability amplitude $f(y)$, with $f(y) = 0$ for $|y| > d$ for some fixed distance $d \ll D$. Use a Huygen's like expression for the wave function on the right simliar to the expresssion used in (a) and (b) above:

$$\psi(x,y) = A \int_{-d}^{d} dy' e^{i\sqrt{x^2+(y-y')^2}\,\frac{p}{\hbar}} f(y').$$

   Making the same approximations as above, show that the probability of finding a particle at the position (D,y), $|\psi(D,y)|^2$, is proportional to the square of the modulus of the Fourier transform of $f(y')$, evaluated at the $p_y = p\frac{y}{D}$. Note that this result for p_y is precisely the y momentum a particle with $p_x \approx p$ would require to reach the point (D,y).



2. Consider a linear operator O acting on a Hilbert space $\mathcal{H}$. Define the *kernel* of O as the subspace of $\mathcal{H}$ which is mapped to zero by O.

   (a) Show that the kernel of O is a vector space.

   Let $O^\dagger$ be the hermitian conjugate of O. (Recall $O^\dagger$ is defined by the relation $(A, O^\dagger B) = (OA, B)$ for all vectors $A, B \in \mathcal{H}$.)

   (b) If the dimension of $\mathcal{H}$ is finite, show that:

   $$\dim(\ker O) = \dim(\ker O^\dagger).$$

   (The rank-nullity theorem.)

   (c) Construct a counter example to this statement if the dimension of $\mathcal{H}$ is infinite.

   (d) Finally, consider a family of linear operators O_α acting on a Hilbert space $\mathcal{H}$. Assume that the operator $O_\alpha O_\alpha^\dagger$ has a complete set of eigenvectors and eigenvalues which depend continuously on the parameter α. Show that the index of O_α,

   $$\mathrm{ind}(O_\alpha) \equiv \dim(\ker O_\alpha) - \dim(\ker O_\alpha^\dagger),$$

   is independent of α. [Hint: consider the operators $O_\alpha O_\alpha^\dagger$ and $O_\alpha^\dagger O_\alpha$ and show that they have one-to-one corresponding, non-zero eigenvalues and eigenvectors, so that any change in $\dim(\ker O_\alpha)$ will be matched by a change in $\dim(\ker O_\alpha^\dagger)$.]