Physics GR6037 Professor Christ
due 10/06/2024

Problem Set #4
(*Hint added to problem 15.*)

11. Consider a free particle of mass m moving in one dimension.

(a) If at $t = 0$ the wave function $\psi(x)$ is the Gaussian

$$\psi(x) = \frac{e^{-\frac{x^2}{4d^2}}}{d^{\frac{1}{2}}(2\pi)^{\frac{1}{4}}} e^{ip_0 x/\hbar}$$

show that at a later time t the wave function becomes:

$$\psi(x,t) = \frac{e^{-i\frac{p_0^2}{2m}t/\hbar}}{(1+\frac{i\hbar t}{2md^2})^{\frac{1}{2}}(d)^{\frac{1}{2}}(2\pi)^{\frac{1}{4}}} e^{ip_0 x/\hbar} \exp\Big(-\frac{(x-\frac{p_0}{m}t)^2}{4d^2(1+\frac{it\hbar}{2md^2})}\Big)$$

(b) Define the Green's function for one-dimensional free particle motion by the expression

$$G(x-x',t) = \langle x|e^{-iHt/\hbar}|x'\rangle,$$

where $H = p^2/2m$. Show that $G(x,t)$ solves the equation:

$$\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}G(x,t) = i\hbar\frac{\partial}{\partial t}G(x,t)$$

and obeys the initial condition $G(x,0) = \delta(x)$.

(c) Show that given $\psi(x,t=0)$ we can use $G(x,t)$ to obtain $\psi(x,t)$ from

$$\psi(x,t) = \int_{-\infty}^{\infty} dx' G(x-x',t)\psi(x',t=0)$$

(d) Find the Green's function $G(x,t)$ as an explicit function of x and t. (Note: G can be obtained by re-purposing the wave function studied in part (a) by changing its normalization and taking the limit $d \to 0$.

12. Consider a particle of charge q and mass m moving in a magnetic field $\vec{B}(\vec{r})$. This system is described by the Hamiltonian:

$$H(\vec{p},\vec{r}) = \frac{\left(\vec{p} - \frac{q}{c}\vec{A}(\vec{r})\right)^2}{2m},$$

where $\vec{A}(\vec{r})$ is a vector potential for the magnetic field $\vec{B} = \vec{\nabla}_{\vec{r}} \times \vec{A}(\vec{r})$.

(a) Find Hamilton's equations expressing $\dot{\vec{p}}$ and $\dot{\vec{r}}$ in terms of $\vec{p}$ and $\vec{r}$.

(b) Combine these equations to show that Newton's law with the standard Lorentz force exerted by a magnetic field on a moving charge is obeyed.

[This is the Hamiltonian mechanics problem mentioned in class.]

13. Find the allowed energies for a particle constrained to move in each of the geometries below. For parts (a) and (c) you should assume that the particle is constrained to move on the specified one-dimension path or two dimensional surface by a strong, three-dimensional potential.

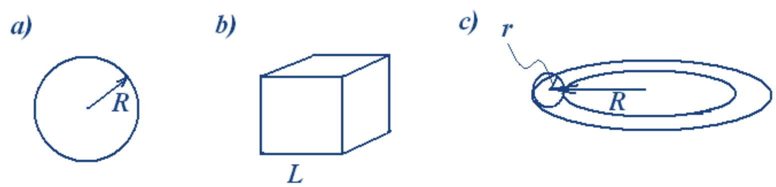
    (a) On a ring of radius R.

    (b) Inside a cube with edges of length L. Assume that the particle acquires an infinite potential energy outside of the cube in order to determine the boundary conditions.

    (c) On the surface of a torus made up of points with the coordinates:

$$\begin{aligned} x &= (R + r\cos\phi)\cos\theta \\ y &= (R + r\cos\phi)\sin\theta \\ z &= r\sin\phi \end{aligned}$$

    with r and R fixed and $0 \le \phi < 2\pi$, $0 \le \theta < 2\pi$. First determine the Schrödinger equation without approximation. Next, when finding the allowed energies you may assume $r \ll R$.



14. A charged particle moving in one dimension is acted on by a simple harmonic force and a constant electric field. The Hamiltonian for such a system is:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 - qEx. \tag{1}$$

    (a) Find the allowed energies and determine the average values of x and p^2 for each eigenstate of H.

    (b) Assume that the system is initially in its ground state. At $t = 0$ the electric field is switched off so rapidly that the particle remains in the old ground state when the field reaches zero. What is the probability that the particle is subsequently found in the ground state of the new system?

15. A particle of mass m is attached to one end of a massless rod of length l. The other end of the rod is attached to a pivot. The rod is free to swing in two dimensions about the pivot P under the influence of gravity (a spherical pendulum). [Hint: Use a regular system of coordinates. For example, the x and y coordinates of the vertical projection of the mass onto the horizontal x-y plane.]

    (a) Find the ground state and the ground state energy in the small angle approximation.

    (b) Continuing to make the small angle approximation, find the next two lowest energy eigenvalues and their corresponding wave functions. How many states corresond to each of these two energies?

    (c) What conditions must be obeyed if the small angle approximation is to be valid for the ground state?