Physics GR6037 Professor Christ

due 11/03/2024

Problem Set #7

*Don't miss the problems on the second page.*

21. Consider the simplified Schrödinger equation:

$$-\frac{d^2}{dx^2} + V(x)\psi(x) = E\psi(x)$$

where $V(x)$ is given by:

$$V(x) = 20e^{-(x-50)^2/48}$$

which describes a barrier penetration problem. Assume that a plane wave with amplitude $e^{-ipx}$ is incident on the barrier from the right, where $p = \sqrt{E}$.

   (a) Find the solution to this equation in the region $0 \le x \le 100$ for E=16 numerically. Plot your results for the real and imaginary parts of $\psi(x)$.

   (b) Find numerical values for the magnitude and phase of the reflected and transmitted waves. Use a step size $Dx = 0.005$ and the leap-frog method implemented in the Schrodinger.ipynb example posted on the web.

For your solution to this problem you need submit only the plot obtained from part a) and the four numbers obtained in part b). [It may be helpful to begin with a working Jupyter notebook which can be downloaded from the course web site, Python, which solves the case $V(x) = 0$ and then to modify this sample program to solve the problem here.]

22. Find the three matrices $(J_i)_{mm'} = \langle j, m|J_i|j, m'\rangle$, $1 \le i \le 3$, explicitly for the case $j = 1$. Verify the commutation relation: $[J_1, J_2] = i\hbar J_3$.

23. Consider a quantum mechanical system with total angular momentum $j = \frac{1}{2}$ (for example a proton at rest with intrinsic spin $j = \frac{1}{2}$). Any state $|\psi\rangle$ of the system can be written as a superposition of the two states $|j = \frac{1}{2}, m = \pm\frac{1}{2}\rangle$:

$$|\psi\rangle = a_+|\frac{1}{2}, \frac{1}{2}\rangle + a_-|\frac{1}{2}, -\frac{1}{2}\rangle.$$

where $a_+$ and $a_-$ are complex numbers.

   (a) If the system is known to be in the state with $J_3 = +\frac{\hbar}{2}$ what is the probability that a measurement of $J_3\cos(\theta) + J_2\sin(\theta)$ will yield the value $+\frac{\hbar}{2}$?

   (b) If $\hat{n}$ is a unit vector pointing in a direction specified by the polar coordinates $\theta$ and $\phi$ find values for the two coefficients $a_+$ and $a_-$ which determine a state $|\psi(\theta, \phi)\rangle$ which obeys

$$\langle\psi(\theta, \phi)|\vec{J}|\psi(\theta, \phi)\rangle = \frac{\hbar}{2}\hat{n}.$$

(c) If the time development of this spin-$\frac{1}{2}$ system is determined by the Hamiltonian

$$H = -\frac{ge}{2mc}\vec{J} \cdot \vec{B}$$

(for example our proton sits in a magnetic field $\vec{B}$), and it is known that at $t = 0$, $J_3 = +\frac{\hbar}{2}$, find the resulting state $|\psi(t)\rangle$ at a later time $t$. (Note, $\vec{B}$ should be assumed to point in a general direction.)

(d) For the case that $\vec{B} = |\vec{B}|\hat{z}$ and the initial state is the state $|\psi(\theta, \phi)\rangle$ found in part (b), find the resulting state at a later time $t$ and show that the expectation value $\langle\psi(t)|\vec{J}|\psi(t)\rangle$ has the same time dependence as a classical magnetic dipole with angular momentum $\vec{J}$ and magnetic moment $\vec{\mu} = \frac{ge}{2mc}\vec{J}$ precessing in the magnetic field $\vec{B}$.

24. Let $|+\frac{1}{2}\rangle$ and $|-\frac{1}{2}\rangle$ be two eigenstates of $J_3$ for a spin-$\frac{1}{2}$ particle with eigenvalues $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$.

(a) Show that for any normalized spin-$\frac{1}{2}$ state $|\psi\rangle$, $\langle\psi|\vec{J}|\psi\rangle\frac{2}{\hbar}$ is a unit vector.

(b) For $j > \frac{1}{2}$, is $\langle\psi|\vec{J}|\psi\rangle\frac{1}{j\hbar}$ a unit vector? Give a proof or a counter example.