Physics GR6037 Professor Christ

due 11/10/2024

Problem Set #8

Corrections made to problems 25e and 27b

The final probem in this set (28.) is an extra problem. It will be graded but the grade will not count toward your final homework score so you should solve it only if you have time.

25. Consider functions $F(x, y, z)$ depending on the three spatial coordinates x, y and z which have the form:

$$F(x,y,z) = \sum_{n_x=0}^{N} \sum_{n_y=0}^{N} \sum_{n_z=0}^{N} c_{n_x n_y n_z} x^{n_x} y^{n_y} z^{n_z}$$

where the coefficients $c_{n_x n_y n_z}$ are complex constants, non-vanishing only when the sum of the three integers $n_x + n_y + n_z = N$, *i.e.* the function F is a homogeneous polynomial of order N.

   (a) Show that if a polynomial with a fixed value of N is viewed as a quantum mechanical wave function and acted on by a rotation operator, it will transform into a second homogenous polynomial which is also of order N. Conclude that the vector space of all such polynomials with a fixed value of N will form a representation of the rotation group.

   (b) If N is even, find that function of the above form with $L^2 = \hbar^2 l(\ell + 1) = 0$.

   (c) If N is odd, find those functions with $\ell = 1$, $m = -1, 0, +1$. (Hint: first consider the case $N = 1$ and examine with the functions $rY_{1,m}(\theta, \phi)$.)

   (d) Find the function with $\ell = N$ and $m = +N$.

   (e) Using the examples above, construct an inductive argument showing that the vector space of polynomials of order N is the same as the vector space constructed from the set of functions $r^N Y_{\ell,m}$, for all ℓ obeying $0 \le \ell \le N$ where ℓ is even if N is even and ℓ is odd if N is odd. [Hint: It may be useful to show that the dimension of these two vector spaces increase by the same amount when N is increased by 2.]

26. Consider the 3-dimensional harmonic oscillator with

$$H = \frac{\vec{p}^{\,2}}{2m} + \frac{1}{2} m\omega^2 \vec{r}^{\,2}.$$

   (a) Find the allowed energies and corresponding multiplicities for this Hamiltonian.

   (b) Observe that the eigenstates of H can be chosen as eigenstates of L^2 and L_z as well. What values of ℓ and m, with what multiplicities are possible for states with a given energy eigenvalue E?
   [Hint: this can be answered using the your solution to problem 25 above. This might be done by extracting from the product of hermite polynomials $h_{n_1}(x_1)h_{n_2}(x_2)h_{n_3}(x_3)$ in each energy eigenstate that term which contain the largest power of each of its three arguments. Such a relation would provide a 1:1 connection between the homogeneous polynomials of problem 25 and the energy eigenstates of this problem.]

   (c) Determine the form of the radial wave function for those states with $E = \hbar\omega(N + \frac{3}{2})$ and $\ell = N$.

27. A spin-1/2 particle with magnetic moment $\vec{\mu} = \gamma \vec{s}$ is placed in a magnetic field $\vec{B}_0 = B_0 \hat{z}$ in the z-direction. Use eigenstates $| \pm \frac{1}{2}\rangle$ of s_z with eigenvalues $\pm\hbar/2$.

(a) For an initial state $|\psi(0)\rangle = a_+ | + \frac{1}{2}\rangle + a_- | - \frac{1}{2}\rangle$, find the resulting state $|\psi(t)\rangle$ at a later time t.

(b) Next a time-dependent magnetic field

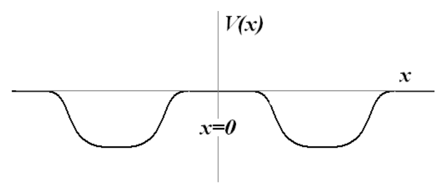
$$\vec{B}_1(t) = B_1 \hat{x} \cos(\omega t) - B_1 \hat{y} \sin(\omega t) \tag{1}$$

is added to the original field $\vec{B}_0$. Solve the resulting time-dependent Schrödinger equation for these combined magnetic fields by working in a "rotating" frame, deriving and solving the Schrödinger equation obeyed by the transformed state:

$$|\psi'(t)\rangle = e^{-i\omega t \hat{z}\cdot\vec{s}/\hbar} |\psi(t)\rangle. \tag{2}$$

[Note: while it offers insight to view this as a transformation to a rotating frame, Eq. (2) can be viewed as a simple mathematical transformation which makes the problem easy to solve without any need for physical intuition.]

(c) Describe the resulting motion of the particle's spin as ω is varied through the Larmor precession frequency $\omega_L = \gamma B_0$.



28. **Extra problem:** Consider a one-dimensional potential $V(x)$ of the form shown, symmetric about $x = 0$: $V(x) = V(-x)$.

(a) Prove that the eigenstates of the Hamiltonian

$$H = \frac{\partial^2}{\partial x^2} \frac{\hbar^2}{2m} + V(x)$$

can be chosen to obey $\psi(x) = \pm\psi(-x)$.

(b) If the two potential wells are far separated then the normalizable eigenstates are approximately the superpositions:

$$\psi_{n,\pm}(x) = \phi_n^L(x) \pm \phi_n^R(x)$$

where $\phi_n^L(x) = \phi_n^R(-x)$ is a normalizable eigenfunction for the potential

$$V'(x) = \begin{cases} V(x) & x < 0 \\ 0 & 0 < x. \end{cases}$$

Use the WKB approximation to find the difference in energy of $\psi_{n,+}$ and $\psi_{n,-}$ for the case of a large but finite separation between the wells.

(c) Does the symmetric or anti-symmetric solution have the lower energy?

[Hint: This problem is less complex than the α decay problem worked in class but does require a careful setup to control the ambiguities associated with the WKB approximation. You might work in the region $x > 0$ and identify two solution with exponentially growing or falling behavior to the left of the smallest $x > 0$ turning point. You might then write the normalizable solution $\psi_{E,\pm}(x)$ as a combination of these two states based on their behavior in the classically allowed region for $x > 0$.]