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Coherent simple harmonic oscillator states

1 The simple harmonic oscillator

Recall the details of the quantum mechancial simple harmonic oscillator (SHO):

$$
\begin{aligned}
H &= \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 && (1) \\
&= \hbar\omega \left(a^\dagger a + \frac{1}{2} \right) && (2)
\end{aligned}
$$

where

$$
a = \sqrt{\frac{m\omega}{2\hbar}} x + i \sqrt{\frac{1}{2\hbar m\omega}} p. \qquad (3)
$$

We can use the operator $a^\dagger$ to construct all of the normalized eigenstates of H:

$$
|n\rangle = \frac{1}{\sqrt{n!}} \left(a^\dagger \right)^n |0\rangle. \qquad (4)
$$

where the state $|0\rangle$ is anihilated by a:

$$
a|0\rangle = 0 \qquad (5)
$$

if we allow 0 to also represent the zero vector in the SHO Hilbert space of states. We can apply H to $|n\rangle$ and obtain:

$$
H|n\rangle = \hbar\omega \left(n + \frac{1}{2} \right) |n\rangle. \qquad (6)
$$

Finally we can easily work out what the wave functions are for the SHO eigenstates:

$$
\begin{aligned}
0 &= \langle x|a|0\rangle && (7) \\
&= \langle x| \left\{ \sqrt{\frac{m\omega}{2\hbar}} x + i \sqrt{\frac{1}{2\hbar m\omega}} p \right\} |0\rangle && (8) \\
&= \sqrt{\frac{\hbar}{2m\omega}} \left\{ \frac{m\omega}{\hbar} x + \frac{d}{dx} \right\} \langle x|0\rangle && (9)
\end{aligned}
$$

which is solved by:

$$
\langle x|0\rangle = \left(\frac{m\omega}{\hbar\pi} \right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar} x^2}. \qquad (10)
$$

where the normalization factor in front has been chosen so that the state$|0\rangle$ has norm 1.

Finally we can find the wave function of for the general state $|n\rangle$:

$$\begin{aligned}
\langle x|n\rangle &= \langle x|\frac{\left(a^\dagger\right)^n}{\sqrt{n!}}|0\rangle && (11)\\
&= \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}\frac{1}{\sqrt{n!}}\left\{\frac{m\omega}{2\hbar}x-\sqrt{\frac{\hbar}{2m\omega}}\frac{d}{dx}\right\}^n e^{-\frac{m\omega}{2\hbar}x^2} && (12)\\
&= \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}\frac{1}{2^{\frac{n}{2}}\sqrt{n!}}\left\{z-\frac{d}{dz}\right\}^n e^{-\frac{1}{2}z^2} && (13)\\
&= \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}\frac{1}{2^{\frac{n}{2}}\sqrt{n!}}\left\{e^{\frac{1}{2}z^2}\left[-\frac{d}{dz}\right]e^{-\frac{1}{2}z^2}\right\}^n e^{-\frac{1}{2}z^2} && (14)\\
&= \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}\frac{1}{2^{\frac{n}{2}}\sqrt{n!}}\left\{e^{\frac{1}{2}z^2}\left[-\frac{d}{dz}\right]^n e^{-\frac{1}{2}z^2}\right\} e^{-\frac{1}{2}z^2} && (15)\\
&= \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}\frac{e^{-\frac{1}{2}z^2}}{2^{\frac{n}{2}}\sqrt{n!}}\left\{e^{z^2}\left[-\frac{d}{dz}\right]^n e^{-z^2}\right\} && (16)\\
&= \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}\frac{e^{-\frac{1}{2}z^2}}{2^{\frac{n}{2}}\sqrt{n!}}h_n(z) && (17)
\end{aligned}$$

In Eq. (13) we have simplified things by introducing the variable $z=\sqrt{m\omega/\hbar}\,x$ while in Eq. (15) we have written out the product of the n factors in curly brackets and recognnized that the two factors of $\exp(\pm\frac{1}{2}z^2)$ cancel in each of the $n-1$ products. Lastly, the Hermite polynomials $h_n(z)$:

$$h_n(z)=e^{z^2}\left[-\frac{d}{dz}\right]^n e^{-z^2} \tag{18}$$

are introduced (and defined) in Eq. (17).

2 Coherent states

Coherent states are superpositions of energy eigenstates which show close to classical behavior and yet are surprisingly simple. The coherent state $|A\rangle$ is defined as an eigenstate of the lowering operator with complex eigenvalue A:

$$a|A\rangle = A|A\rangle. \tag{19}$$

This equation can be easily used to express the state $|A\rangle$ as a sum over eigenstates of the Hamiltonian:

$$\begin{aligned}
|A\rangle &= \sum_{n=0}^{\infty}\frac{A^n}{\sqrt{n!}}|n\rangle e^{-\frac{1}{2}|A|^2} && (20)\\
&= \sum_{n=0}^{\infty}\frac{A^n}{n!}\left(a^\dagger\right)^n|0\rangle e^{-\frac{1}{2}|A|^2}. && (21)
\end{aligned}$$

That the state defined in this equation actually obeys Eq. (19) can be seen by applying the lowering operator a to both sides and remembering that $a\,|n\rangle = \sqrt{n}|n-1\rangle$. The added factor of $\exp -\frac{1}{2}|A|^2$ in these two equation is introduced to normalize $|A\rangle$: $\langle A|A\rangle = 1$

Equation (20) serves to give a precise definition of the state $|A\rangle$, including its phase and can also be used to evaluate its time dependence:

$$\begin{aligned} e^{-i\frac{1}{\hbar}Ht}|A\rangle &= \sum_{n=0}^{\infty} \frac{A^n}{\sqrt{n!}} e^{-i\frac{1}{\hbar}Ht}|n\rangle e^{-\frac{1}{2}|A|^2} && (22)\\ &= \sum_{n=0}^{\infty} \frac{A^n}{\sqrt{n!}} e^{-i\omega(n+\frac{1}{2})t}|n\rangle e^{-\frac{1}{2}|A|^2} && (23)\\ &= e^{-i\frac{1}{2}\omega t}\sum_{n=0}^{\infty} \frac{(Ae^{-i\omega t})^n}{\sqrt{n!}}|n\rangle e^{-\frac{1}{2}|A|^2} && (24)\\ &= e^{-i\frac{1}{2}\omega t}|Ae^{-i\omega t}\rangle, && (25) \end{aligned}$$

so that the time evolution involves adding a simple time-dependent phase to the complex eigenvalue A!

We can find the wave function $\langle x|A\rangle$ if we substitued Eq. (16) into Eq. (20):

$$\begin{aligned} \langle x|A\rangle &= \sum_{n=0}^{\infty} \frac{A^n}{\sqrt{n!}} \langle x|n\rangle e^{-\frac{1}{2}|A|^2} && (26)\\ &= \sum_{n=0}^{\infty} \frac{A^n}{\sqrt{n!}} \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} \frac{e^{-\frac{1}{2}z^2}}{2^{\frac{n}{2}}\sqrt{n!}} \left\{ e^{z^2}\left[-\frac{d}{dz}\right]^n e^{-z^2}\right\} e^{-\frac{1}{2}|A|^2} && (27)\\ &= \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}z^2}\left\{ e^{z^2} e^{-(z-\frac{1}{\sqrt{2}}A)^2}\right\} e^{-\frac{1}{2}|A|^2} && (28)\\ &= \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}(z-\sqrt{2}A)^2} e^{-\frac{1}{2}(A^2-|A|^2)}. && (29) \end{aligned}$$

If we consider a time-dependent wave function by replacing A by $A\exp(-i\omega t)$ then Eq. (29) will provide a wave function which is a periodic function of the time. The real part of the exponent will correspond to a Gaussian wave packet oscillating back and forth in a classical fashion. The periodic time dependence thus shows no dispersion, very diffferent from what is seen for a free particle described by a Gaussian wave function where the Gaussian spreads out with increasing time!