Physics GR6037 | Professor Christ
due 09/14/2025

Problem Set #1

**Don't miss the second problem on the next page.**

1. Consider a two-dimensional problem in which a beam of particles moves to the right in the $+x$ direction with momentum $p$. The particles in the beam scatter from an array of $N+1$ atoms arranged along the $y$ axis with coordinates $(x = 0, y = jd)$ where $j$ is an integer in the range $-N/2 \le j \le N/2$, $d$ is the separation of the atoms in the vertical direction and $N$ is even. Assume that each of the scattered particles can be represented by a wave which is the superposition of the $N+1$ circular waves:

$$\psi(x,y) = A \sum_{j=-N/2}^{N/2} e^{i\sqrt{x^2+(y-jd)^2}\,\frac{p}{\hbar}}.$$

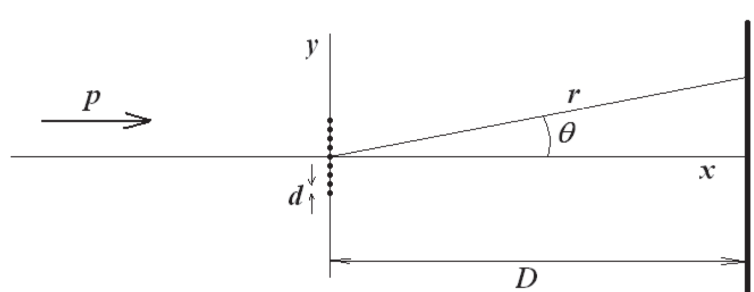
   (a) In the limit that the distance $r = \sqrt{x^2+y^2}$ is large but $\sin(\theta) = y/r$ is small and fixed, expand the exponents in the above expression for $\psi(x,y)$, retaining the term proportional to $r$ and the term which is a constant as $r$ becomes large.

   (b) Using this approximation, perform the sum over $j$ and determine the resulting interference pattern that would would be seen on a screen a large distance $D$ from the atoms.

   (c) Finally replace the array of atoms by a vertical, attenuating screen placed along the $y$ axis. Assume that the incident wave passes through this screen with probability amplitude $f(y)$, with $f(y) = 0$ for $|y| > d$ for some fixed distance $d \ll D$. Use a Huygen's like expression for the wave function on the right simliar to the expresssion used in (a) and (b) above:

$$\psi(x,y) = A \int_{-d}^{d} dy' e^{i\sqrt{x^2+(y-y')^2}\,\frac{p}{\hbar}} f(y').$$

   Making the same approximations as above, show that the probability of finding a particle at the position $(D, y)$, $|\psi(D,y)|^2$, is proportional to the square of the modulus of the Fourier transform of $f(y')$, evaluated at the $p_y = p\frac{y}{D}$. Note that this result for $p_y$ is precisely the y momentum a particle with $p_x \approx p$ would require to reach the point $(D, y)$.



2. Consider a particle moving in one dimension and located at the position $x$ where $x$ lies in the interval $[0, L]$. If $\psi(x)$ is the particle's wave function, assume that $\psi(x)$ obeys periodic boundary conditions:

$$\psi(0) = \psi(L) \quad \text{and} \quad \left.\frac{d}{dx}\psi(x)\right|_{x=0} = \left.\frac{d}{dx}\psi(x)\right|_{x=L}$$

   (a) Consider the wave functions

$$\phi_n(x) = \frac{1}{\sqrt{L}} e^{ip_n x/\hbar}$$

   where $p_n = 2\pi\hbar n/L$ and $n \in \mathbb{Z}$ is an integer. Show that the functions $\phi_n(x)$ obey:

$$\int_0^L dx\, \phi_{n'}^*(x)\phi_n(x) = \begin{cases} 1 & \text{for } n' = n \\ 0 & \text{for } n' \neq n \end{cases}$$

   (b) If the wave function $\psi(x)$ can be written as the sum

$$\psi(x) = \sum_{n\in\mathbb{Z}} a_n \phi_n(x)$$

   where the coefficients $\{a_n\}_{-\infty<n<\infty}$ are complex numbers, derive a formula that can be used to determine the coeficient $a_n$ from the function $\psi(x)$.

   (c) What condition must the coefficients $a_n$ obey if the function $\psi(x)$ is normalized to 1?

3. Consider a linear operator $O$ acting on a Hilbert space $\mathcal{H}$. Define the *kernel* of $O$ as the subspace of $\mathcal{H}$ which is mapped to the zero vector by $O$.

   (a) Show that the kernel of $O$ is a vector space.

   Let $O^\dagger$ be the hermitian conjugate of $O$. (Recall $O^\dagger$ is defined by the relation $(A, O^\dagger B) = (OA, B)$ for all vectors $A, B \in \mathcal{H}$.)

   (b) If the dimension of $\mathcal{H}$ is finite, show that:

$$\dim(\ker O) = \dim(\ker O^\dagger).$$

   (The rank-nullity theorem.)

   (c) Construct a counter example to this statement if the dimension of $\mathcal{H}$ is infinite.