Physics GR6037 Professor Christ

due 12/7/2025

Problem Set #11

41. Calculate the Clebsch-Gordon coefficients $(j, m|j_1, m_1; j_2, m_2)$ for
    (a) $j_1 = 1/2$, $j_2 = 1$
    (b) $j_1 = 1$, $j_2 = 1$
    (c) $j_1 = 1/2$, $j_2 =$ arbitrary.
    (d) For the $J_1 = 1$, $J_2 = 1$ case above, consider the 9 possible products $\{A_j B_j\}_{1\le i,j\le 3}$ formed from the Cartesian components of two three-dimensional vectors $\vec{A} = (A_1, A_2, A_3)$ and $\vec{B} = (B_1, B_2, B_3)$. Group these 9 possible products into irreducible representations of the rotation group and recognize from the number of independent elements in each group to which representation found in part (b) each of these groups corresponds. [These groups can be formed of combinations of the products $\{A_j B_j\}_{1\le i,j\le 3}$ that give a known behavior (for example, the single combination $A_1B_1 + A_2B_2 + A_3B_3$ corresponds to the j=0 representation) or to have a property that is not changed by rotation (for example where the combination is symmetric or anti-symmetric under the exchange of A and B.]

42. Two spin-1 particles interact through the Hamiltonian

$$H = g\vec{S}_1 \cdot \vec{S}_2$$

    (a) What are the allowed energies and corresponding eigenstates for this Hamiltonian? Express the eigenstates in terms of the standard basis of states with definite values of $(S_1)_z$ and $(S_2)_z$.
    (b) If initially the system is in the state with $(S_1)_z = (S_2)_z = 0$, what is the probability of finding $(S_1)_z = \hbar$ at a later time t?

43. The nucleus of a deuterium atom has an electric quadrupole moment described by the operator:

$$Q_{i,j} = Q\left(I_i I_j + I_j I_i - \frac{4}{3}\hbar^2\delta_{i,j}\right)$$

where $\vec{I}$ is the angular momentum operator of the spin-1 nucleus and Q is a constant. This quadrupole moment produces an electrostatic potential:

$$V(\vec{r}) = \sum_{ij}(\hat{r}_i\hat{r}_j - \frac{1}{3}\delta_{i,j})Q_{i,j}\frac{1}{r^3}.$$

You should ignore the effects of the nuclear magnetic moment and Lamb shift but include the effects of fine structure when answering (a), (b) and (c) below.

    (a) Describe the degeneracies present among the deuterium energy eigenstates and deduce which states should be used for a first-order, perturbative treatment of the effects of $V(\vec{r})$ on the atom which deals properly with these degeneracies.

(b) Exploiting parts 41b and 41d above, show that a symmetric, traceless second rank tensor has five independent components and conclude that Q_{ij} are the elements of a representation of $SU(2)$ with $j = 2$.

(c) Using the states chosen in (a), compute the energy levels of the deuterium atom to first order in Q as follows:

i. Use the Wigner-Eckart theorem to show that

$$\langle j, m|\hat{r}_i\hat{r}_j - \frac{1}{3}\delta_{i,j}|j, m'\rangle = c\langle j, m|J_iJ_j + J_jJ_i - \frac{2}{3}j(j+1)\hbar^2\delta_{i,j}|j, m'\rangle.$$

Here $|j, m\rangle$ are the usual Hydrogen atom fine structure eigenstates with total orbital plus spin angular momentum quantum numbers $\vec{J}^2 = \hbar^2 j(j+1)$ and $J_z = \hbar m$.

ii. To determine c note that if the two second-rank tensor operators above are called O^r_{ij} and O^J_{ij}, then c can be determined by comparing the matrix elements of $J_iO^r_{ij}J_j$ and $J_iO^J_{ij}J_j$. Here i and j are summed from 1 to 3.

iii. The combination $J_iO^J_{ij}J_j$ is not difficult to evaluate. While $J_iO^r_{ij}J_j$ looks more difficult, it simpfies quickly when $\vec{J}$ is replaced by $\vec{L} + \frac{\hbar}{2}\vec{\sigma}$ and the properties of the Pauli matrices exploited.

iv. You may use the relation:

$$\langle n, l|\frac{1}{r^3}|n, l\rangle = \frac{1}{a_0^3 n^3 l(l+1/2)(l+1)}.$$

(d) Make a rough numerical comparison of the splitting obtained in (b) and the usual hyperfine separation.