Physics GR6037 Professor Christ

due 9/28/2025

Problem Set #3

8. Consider two self-adjoint operators A and B. Assume that spectrum of both A and B is discrete with a corresponding complete sets of eigenvectors:

$$\begin{aligned} A|n\rangle_A &= a_n|n\rangle_A \\ B|n\rangle_B &= b_n|n\rangle_B. \end{aligned}$$

If A and B have a vanishing commutator, $[A, B] = 0$, show that A and B can be simultaneously diagonalized, *i.e.* there exists a common set of eigenvectors for both operators for each of the following situations of increasing difficulty:

   (a) The spectrum of A and B is non-degenerate, *i.e.* for each eigenvalue of A there exists only a single eigenvector of A with that eigenvalue.

   (b) All degeneracies are finite dimensional, *i.e.* for each eigenvalue of A there exists only a finite number of independent eigenvectors of A with that eigenvalue. [Hint: here you can use the result of problem 5 from Problem Set #2 to diagonalize a finite, hermitian matrix.]

   (c) The case were infinite numbers of eigenvectors with the same eigenvalue are allowed. [Hint: In contrast with 6.b) we can not use problem 5 to argue that we can diagonalize an infinite-dimensional hermitian matrix. This difficulty might be avoided by applying the projection operator $P^B_{b_n}$ onto the known, discrete eigenvectors of A. Here $P^B_{b_n}$ projects onto the infinite-dimensional subspace spanned by the set of eigenvectors $\{|b_n, j\rangle_B\}_j$ of B with the same eigenvalue b_n.]

9. Prove the Schwartz inequality:

$$|\langle A|B\rangle|^2 \leq \langle A|A\rangle\langle B|B\rangle.$$

[Hint: you might use a method similar to our proof of the uncertainty relation from the commutator $[x, p] = i\hbar$ by examining the square of the norm of the vector $|A\rangle + \alpha|B\rangle$ as a function of the complex variable α and requiring that it aways be non-negative.]

10. Using very approximate uncertainty principle arguments, determine:

   (a) The size of the hydrogen atom in terms of the electronic charge and mass.

   (b) The spread of α-particle energies observed in the decay:

$$\mathrm{Po}^{213} \to \mathrm{Pb}^{209} + \alpha$$

   if the Po^{213} nucleus has a lifetime of 4×10^{-6} seconds.

   (c) The maximum length of time a pencil can be balanced on its point. (Assume the pencil has a perfect point and that it is no longer balanced when it makes a noticible angle, perhaps 10-15°, with the vertical.)

11. If $\Delta x\, \Delta p = \frac{\hbar}{2}$ for a state ψ, show that $\psi(x)$ must be a Gaussian wave function. [Hint: This can be shown by returning to the derivation of the uncertainty relation given in class and examining the implications for $\psi(x)$ of the limiting case $\Delta x\, \Delta p = \frac{\hbar}{2}$.]

12. If the commutator $[A, B]$ itself commutes with both A and B, *i.e.* $[A, [A, B]] = [B, [A, B]] = 0$, show that

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}. \tag{1}$$

[Hint: You might introduce a parameter α into the exponents by multiplying A and B by α. Show that Eq. (1) can be rewritten $e^{-A} e^{A+B} e^{-B} = e^{-\frac{1}{2}[A,B]}$ and derive a differential equation in the variable α obeyed by the left-hand side of this equation which you can solve.]