Physics GR6037 Professor Christ
due 11/02/2025

Problem Set #7

25. Consider the simplified Schrödinger equation: $\left\{-\frac{d^2}{dx^2} + V(x)\right\}\psi(x) = E\psi(x)$ where $V(x)$ is given by: $V(x) = 20e^{-(x-50)^2/48}$ which describes a barrier penetration problem. Assume that a plane wave with amplitude e^{-ipx} is incident on the barrier from the right, where $p = \sqrt{E}$.

    (a) Find the solution to this equation in the region $0 \le x \le 100$ for E=16 numerically. Plot your results for the real and imaginary parts of $\psi(x)$.

    (b) Find numerical values for the magnitude and phase of the reflected and transmitted waves. Use a step size $Dx = 0.005$ and the leap-frog method implemented in the Schrodinger.ipynb example posted on the web.

    For your solution to this problem you need submit only the plot obtained from part a) and the four numbers obtained in part b). [It may be helpful to begin with a working Jupyter notebook which can be downloaded from the course web site, Python, which solves the case $V(x) = 0$ and then to modify this sample program to solve the problem here.]

26. Consider functions $F(x, y, z)$ depending on the three spatial coordinates x, y and z which have the form:

$$F(x, y, z) = \sum_{n_x=0}^{N}\sum_{n_y=0}^{N}\sum_{n_z=0}^{N} c_{n_x n_y n_z} x^{n_x} y^{n_y} z^{n_z}$$

    where the coefficients $c_{n_x n_y n_z}$ are complex constants, non-vanishing only when the sum of the three integers $n_x + n_y + n_z = N$, *i.e.* the function F is a homogeneous polynomial of order N.

    (a) Show that if a polynomial with a fixed value of N is viewed as a quantum mechanical wave function and acted on by a rotation operator, it will transform into a second homogenous polynomial which is also of order N. Conclude that the vector space of all such polynomials with a fixed value of N will form a representation of the rotation group.

    (b) If N is even, find that function of the above form with $L^2 = \hbar^2 l(\ell + 1) = 0$.

    (c) If N is odd, find those functions with $\ell = 1$, $m = -1, 0, +1$. (Hint: first consider the case $N = 1$ and examine with the functions $rY_{1,m}(\theta, \phi)$.)

    (d) Find the function with $\ell = N$ and $m = +N$.

    (e) Using the examples above, construct an inductive argument showing that the vector space of polynomials of order N is the same as the vector space constructed from the set of functions $r^N Y_{\ell,m}$, for all ℓ obeying $0 \le \ell \le N$ where ℓ is even if N is even and ℓ is odd if N is odd. [Hint: It may be useful to show that the dimension of these two vector spaces increase by the same amount when N is increased by 2.]

27. Consider the 3-dimensional harmonic oscillator with

$$H = \frac{\vec{p}^{\,2}}{2m} + \frac{1}{2}m\omega^2\vec{r}^{\,2}.$$

(a) Find the allowed energies and corresponding multiplicities for this Hamiltonian.

(b) Observe that the eigenstates of H can be chosen as eigenstates of L^2 and L_z as well. What values of ℓ and m, with what multiplicities are possible for states with a given energy eigenvalue E?
[Hint: this can be answered using the your solution to problem 25 above. This might be done by extracting from the product of hermite polynomials $h_{n_1}(x_1)h_{n_2}(x_2)h_{n_3}(x_3)$ in each energy eigenstate that term which contain the largest power of each of its three arguments. Such a relation would provide a 1:1 connection between the homogeneous polynomials of problem 26 and the energy eigenstates of this problem.]

(c) Determine the form of the radial wave function for those states with $E = \hbar\omega(N + \frac{3}{2})$ and $\ell = N$.

28. Let $|+\frac{1}{2}\rangle$ and $|-\frac{1}{2}\rangle$ be two eigenstates of J_3 for a spin-$\frac{1}{2}$ particle with eigenvalues $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$.

(a) Show that for any normalized spin-$\frac{1}{2}$ state $|\psi\rangle$, $\langle\psi|\vec{J}|\psi\rangle\frac{2}{\hbar}$ is a unit vector.

(b) For $j > \frac{1}{2}$, is $\langle\psi|\vec{J}|\psi\rangle\frac{1}{j\hbar}$ a unit vector? Give a proof or a counter example.