Physics GR6037 Professor Christ

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Representations of the Rotation Group and their Generators

We identify a rotation by the pair $(\hat{n}, \theta)$ where the unit vector $\hat{n}$ is the direction of the rotation and $0 \le \theta \le \pi$ is the clockwise angle through which the rotation is performed. The rotations $(\pm\hat{n}, \pi)$ are equivalent. We will assume that for each rotation, $(\hat{n}, \theta)$, there exists a corresponding unitary operator $R(\hat{n}, \theta)$ which acts on the states in our quantum mechanical Hilbert space and actively rotates the states:

$$|\psi_{\text{rotated}}\rangle = R(\hat{n}, \theta)|\psi\rangle \tag{1}$$

For $i = 1, 2$ and 3, we define:

$$J_i = i\hbar \frac{d}{d\theta} R(\hat{e}_i, \theta)|_{\theta=0}. \tag{2}$$

Just as with translations, we can differentiate the relation:

$$R(\hat{n}, \theta)R(\hat{n}, \theta)^\dagger = I \tag{3}$$

with respect to θ and set $\theta = 0$ to obtain:

$$\frac{d}{d\theta} R(\hat{e}_i, \theta)|_{\theta=0} + \frac{d}{d\theta} R(\hat{e}_i, \theta)^\dagger|_{\theta=0} = 0, \tag{4}$$

which implies that J_i defined above is a hermitian operator. The three operators J_i are the generators of the rotation group for the representation R.

If we consider rotations about a fixed direction, for example one of the directions $\hat{e}_i$, then the additive property of rotations about a fixed direction implies:

$$R(\hat{e}_i, \theta)R(\hat{e}_i, \theta') = R(\hat{e}_i, \theta + \theta'). \tag{5}$$

As we did for translations we can use the definition of the generators given in Eq. (2) to derive a general formula for a rotation about $\hat{e}_i$ through an angle

θ which is not necessarily small:

$$\begin{aligned}
\frac{d}{d\theta}R(\hat{e}_i,\theta) &= \frac{d}{d\theta}R(\hat{e}_i,\theta-\theta')R(\hat{e}_i,\theta') && (6)\\
&= \frac{d}{d\theta}\left[R(\hat{e}_i,\theta-\theta')R(\hat{e}_i,\theta')\right]_{\theta'=\theta} && (7)\\
&= -\frac{i}{\hbar}J_iR(\hat{e}_i,\theta), && (8)
\end{aligned}$$

a familiar equation that can be solved by:

$$R(\hat{e}_i,\theta) = e^{-\frac{i}{\hbar}J_i\theta} \qquad (9)$$

Given the ability to Taylor expand a rotation about a particalar coordinate axis $\hat{e}_i$ in θ we can derive a more general relation for rotation through an arbitray direction $\hat{n}$ through a small angle θ. We start using the rotation of arbitrary vectors in 3-d as an example which fixes how rotations must combine. Note, a small rotation $(\hat{n},\theta)$ of the vector $\vec{v}$ is easily written:

$$\vec{v} \to \vec{v} + \theta\hat{n}\times\vec{v} + O(\theta^2). \qquad (10)$$

Using this concrete formula it is easy to see that for small θ the rotation $(\hat{n},\theta)$ can be written at the product of three rotations:

$$(1+\theta\hat{n}\times)\vec{v} \approx (1+\theta n_1\hat{e}_1\times)(1+\theta n_2\hat{e}_2\times)(1+\theta n_3\hat{e}_3\times)\vec{v} + O(\theta^2) \qquad (11)$$

Since our R's must obey the same multiplication laws we conclude that:

$$\begin{aligned}
R(\hat{n},\theta) &\approx R(\hat{e}_1,\theta n_1)R(\hat{e}_2,\theta n_2)R(\hat{e}_3,\theta n_3) && (12)\\
&\approx (I-i\theta n_1J_1/\hbar)(I-i\theta n_2J_2/\hbar)(I-i\theta n_3J_3/\hbar) && (13)\\
&\approx I-i\theta n_1J_1/\hbar-i\theta n_2J_2/\hbar-i\theta n_3J_3/\hbar) && (14)\\
&\approx I-i\theta\hat{n}\cdot\vec{J}/\hbar && (15)
\end{aligned}$$

This implies that

$$\hat{n}\cdot\vec{J} = i\hbar\frac{d}{d\theta}R(\hat{n},\theta)|_{\theta=0}. \qquad (16)$$

As in Eqs. (6), (7) and (8) we can derive the equivalent of Eq. (9):

$$R(\hat{n},\theta) = e^{-\frac{i}{\hbar}\hat{n}\cdot\vec{J}\theta} \qquad (17)$$

Next we study how the generators themselves rotate. Consider the matrix element of an observable O between the states $|A\rangle$ and $|B\rangle$. Such a matrix element will typically change if we rotate the states: $|A\rangle$ and $|B\rangle$:

$$\langle A|O|B\rangle \to \langle A_{\mathrm{rotated}}|O|B_{\mathrm{rotated}}\rangle = \langle A|R(\hat{n},\theta)^{\dagger} O R(\hat{n},\theta)|B\rangle. \tag{18}$$

Here the right-most expression can be viewed as evaluating the operator O between rotated states or evaluating the *rotated operator* $R(\hat{n},\theta)^{\dagger} O R(\hat{n},\theta)$ between the original states. Thus, it is completely equivalent to think of rotating the operators or rotating the states and we might consider rotating a rotation operator:

$$R(\hat{n},\theta)R(\hat{n}',\theta')R(\hat{n},-\theta) \equiv R(\hat{n}'',\theta'') \tag{19}$$

Since we understand 3-d rotations, given $(\hat{n},\theta)$ and $(\hat{n}',\theta')$ we could, in principle, figure out $(\hat{n}'',\theta'')$. The resulting $(\hat{n}'',\theta'')$ is determined entirely by the rotations $(\hat{n},\theta)$ and $(\hat{n}',\theta')$ and will be the same for all representations. As we will see this is all that is needed to completely determine the operators J_i.

To understand the effects of a rotation on J_i, we should consider the case where the middle angle θ' is small and Taylor expand the rotation $R(\hat{n}',\theta')$:

$$R(\hat{n},\theta)\left(I - i\theta'\hat{n}'\cdot\vec{J}/\hbar\right)R(\hat{n},-\theta) = I - i\theta''\hat{n}''\cdot\vec{J}/\hbar. \tag{20}$$

Here we have used the fact that as the middle rotation approaches the identity, $R(\hat{n}',\theta') \to I$ the right-hand side must also so that θ'' must also become small.

This equation tell us that the combination $\theta''\hat{n}''$ must be a linear function of product of the components of $\hat{n}'$ and θ. Therefore we write:

$$\hat{\theta}''n_i'' = \sum_j M_{i,j}(\hat{n},\theta)n_j'\theta'. \tag{21}$$

Substituting this equation into the one before and choosing $\hat{n}' = \hat{e}_j$, we find:

$$R(\hat{n},\theta)J_jR(\hat{n},-\theta) = \sum_i M_{i,j}(\hat{n},\theta)J_i. \tag{22}$$

This result that the three operators J_i transform under rotations by the action of the 3×3 matrix $M(\hat{n},\theta)$ can be described by saying that the the three operators J_i can be viewed as basis vectors of a three-dimensional Hilbert space upon which rotations act. The matrices $M(\hat{n}, \theta)$ are then the rotation matrices for this representation. The special representation where the group operations act on the group generators is called the adjoint representation. Since we know that $\vec{J}$ will ultimately turn out to be the pseudo-vector operator for angular momentum, we recognize that for rotations this adjoint representation is simply the normal 3-dimensional rotation of 3-d vectors.

The last step is to examine the generators, $\mathcal{J}_k$, of the matrices $M(\hat{n}, \theta)$ themselves. We consider θ small and write:

$$M(\hat{n}, \theta) \approx I - i\theta\hat{n} \cdot \vec{\mathcal{J}}/\hbar + O(\theta^2). \tag{23}$$

We can go back and expand the rotation $R(\hat{n}, \theta)$ in Eq. 22 above and derive a simple equation for $\vec{\mathcal{J}}$:

$$(I - i\theta\hat{n} \cdot J/\hbar)J_j(I + i\theta\hat{n} \cdot J/\hbar) = \sum_i \left(I - i\hat{n} \cdot \vec{\mathcal{J}}\theta/\hbar\right)_{ij} J_i. \tag{24}$$

Dropping the common J_j term from each side, we find:

$$-i\theta[\hat{n} \cdot J/\hbar, J_j] = -i\theta \sum_i \hat{n} \cdot \left(\vec{\mathcal{J}}_{ij}/\hbar\right) J_i. \tag{25}$$

If we cancel common factors of $-i\theta/\hbar$ and chose $\hat{n} = \hat{e}_k$ this reduces to:

$$[J_k, J_j] = \sum_i (\mathcal{J}_k)_{ij} J_i. \tag{26}$$

Note, this equation describes how the generators of rotations themselves transform under rotations. This summarizes the properties of the underlying rotation group describing how products of small rotations can result in a related small rotation. The matrix $M_{i,j}(\hat{n}, \theta)$ and its generators $(\mathcal{J}_k)_{ij}$ describe this relation. Because the matrices in a representation follow the same multiplication rules as those in the underlying group, the matrix $M_{i,j}(\hat{n}, \theta)$ and its generators $(\mathcal{J}_k)_{ij}$ must be the same for any representation R that is being examined.

Thus, we can determine $(\mathcal{J}_k)_{ij}$ by considering the combination of simple rotations in three dimensions. Equation 10 gives a simple form for the generator of 3-D rotations:

$$(-iJ_k/\hbar)\vec{v} = \hat{e}_k \times \vec{v} \quad \text{or} \quad J_k\,\vec{v} = i\hbar\,\hat{e}_k \times \vec{v}. \tag{27}$$

This can be substituted into Eq. 26 and the resulting equation applied to the vector $\vec{v}$, giving

$$\begin{aligned}
(i\hbar\hat{e}_k\times)(i\hbar\hat{e}_j\times)\vec{v} - (i\hbar\hat{e}_j\times)(i\hbar\hat{e}_k\times)\vec{v} &= \sum_i (\mathcal{J}_k)_{ij}(i\hbar\hat{e}_i)\times\vec{v} &(28)\\
i\hbar\hat{e}_k\times(\hat{e}_j\times\vec{v}) - i\hbar\hat{e}_j\times(\hat{e}_k\times\vec{v}) &= \sum_i (\mathcal{J}_k)_{ij}\hat{e}_i\times\vec{v} &(29)\\
i\hbar\Big(\hat{e}_j(\hat{e}_k\cdot\vec{v}) - \vec{v}(\hat{e}_k\cdot\hat{e}_j) - \hat{e}_k(\hat{e}_j\cdot\vec{v}) + \vec{v}(\hat{e}_j\cdot\hat{e}_k)\Big) & &(30)\\
&= \sum_i (\mathcal{J}_k)_{ij}\hat{e}_i\times\vec{v}\\
i\hbar\Big(\hat{e}_j(\hat{e}_k\cdot\vec{v}) - \hat{e}_k(\hat{e}_j\cdot\vec{v})\Big) &= \sum_i (\mathcal{J}_k)_{ij}\hat{e}_i\times\vec{v} &(31)\\
i\hbar(\hat{e}_k\times\hat{e}_j)\times\vec{v} &= \sum_i (\mathcal{J}_k)_{ij}\hat{e}_i\times\vec{v} &(32)
\end{aligned}$$

Since $\hat{e}_k\times\hat{e}_j = \sum_i \epsilon_{kji}\hat{e}_i$, we learn from Eq. 32 that

$$(\mathcal{J}_k)_{ij} = i\hbar\epsilon_{kji} \tag{33}$$

and that

$$[J_k, J_j] = \sum_i (\mathcal{J}_k)_{ij}J_i = i\hbar\sum_i \epsilon_{kji}J_i. \tag{34}$$

As was worked out in class, these commutation relations obeyed by the generators of a representation of the rotation group are sufficient to determine all possible finite dimensional respresentations of rotations. The key step in this demonstration was recognizing that two operators, usually taken to be $\vec{J}^2 = J_1^2 + J_2^2 + J_3^2$ and J_3 commute with each other and can therefore be simultaneously diagonalized and that the operator $J_1 + iJ_2$ acts as a raising operator for J_3:

$$[J_3, J_1 + iJ_2] = +\hbar(J_1 + iJ_2.) \tag{35}$$

Just as the case of the simple harmonic oscillator with the equation

$$[H, a^\dagger] = +\hbar\omega a^\dagger, \tag{36}$$

Eq. (35) can be used to show that when acting on an eigenstate of J_3 the operator $J_1 + iJ_2$ creates a new eignestate of J_3 with eigenvalue increased by $\hbar$, which can then be used to determine every possible set of representation matrices J_1, J_2 and J_3,