Physics G6037 Professor Christ
12/01/2025

Wigner 3-j Symbols

Begin by considering states on which two angular momentum operators $\vec{J}^{\,1}$ and $\vec{J}^{\,2}$ are defined: $|j_1, m_1; j_2, m_2\rangle$. As the labels suggest, these states are eigenstates of $(\vec{J^i})^2$ and J_z^i:

$$\begin{aligned}(\vec{J^i})^2|j_1, m_1; j_2, m_2\rangle &= \hbar^2 j_i(j_i+1)|j_1, m_1; j_2, m_2\rangle &(1)\\ J_z^i|j_1, m_1; j_2, m_2\rangle &= \hbar m_i|j_1, m_1; j_2, m_2\rangle &(2)\end{aligned}$$

for $i = 1, 2$. We next introduce the Clebsch-Gordan coefficients that determine how these states can be combined to produce states which are eigenstates of $(\vec{J})^2$ and J_z where $\vec{J} = \vec{J}^{\,1} + \vec{J}^{\,2}$ and

$$\begin{aligned}(\vec{J})^2|j, m; j_1, j_2\rangle &= \hbar^2 j(j+1)|j, m; j_1, j_2\rangle &(3)\\ J_z|j, m; j_1, j_2\rangle &= \hbar m_i|j, m; j_1, j_2\rangle &(4)\end{aligned}$$

These coefficients are defined by:

$$(j, m|j_1, m_1; j_2, m_2) = \langle j, m; j_1, j_2|j_1, m_1; j_2, m_2\rangle. \quad (5)$$

Since by convention the Clebsch-Gordan coefficients are always chosen real the order in which the right and left states are written does not matter:

$$\begin{aligned}\langle j, m; j_1, j_2|j_1, m_1; j_2, m_2\rangle &= \langle j_1, m_1; j_2, m_2|j, m; j_1, j_2\rangle^* \\ &= \langle j_1, m_1; j_2, m_2|j, m; j_1, j_2\rangle. &(6)\end{aligned}$$

Define a general rotation acting on such a $\vec{J}^2$, J_z eigenstate to take the form:

$$\exp(-i\vec{J}\cdot\hat{n}\,\theta/\hbar)|j, m\rangle = \sum_{m'=-j}^{j} \mathcal{D}^j_{m',m}(\hat{n}, \theta)|j, m'\rangle \quad (7)$$

If we insert the product of operators $\exp(+i\vec{J}\cdot\hat{n}\,\theta/\hbar)\exp(-i\vec{J}\cdot\hat{n}\,\theta/\hbar)$ in the matrix element in Eq. 5 we find that the Clebsch-Gordan coefficients obey the following relation:

$$\begin{aligned}(j, m|j_1, m_1; j_2, m_2) &= \sum_{m'=-j}^{j}\sum_{m_1'=-j_1}^{j_1}\sum_{m_2'=-j_2}^{j_2} \mathcal{D}^j_{m',m}(\hat{n}, \theta)^* \\ &\quad\cdot\mathcal{D}^{j_1}_{m_1',m_1}(\hat{n}, \theta)\mathcal{D}^{j_2}_{m_2',m_2}(\hat{n}, \theta)(j, m'|j_1, m_1'; j_2, m_2'). &(8)\end{aligned}$$

We can remove the complex conjugate on the first $\mathcal{D}^j_{m',m}(\hat{n},\theta)^*$ factor by recognizing that in our usual conventions the elements of the matrices J_x and J_z are real while those of J_y are imaginary allowing us to write:

$$(e^{-i\vec{J}\cdot\hat{n}\theta/\hbar})^* = e^{-i(-J_x n_x + J_y n_y - J_z n_z)\theta/\hbar} = e^{+i\pi J_y/\hbar} e^{-i\vec{J}\cdot\hat{n}\theta/\hbar} e^{-i\pi J_y/\hbar}. \tag{9}$$

By commuting $e^{-i\pi J_y/\hbar}$ with J_z and $J_x \pm iJ_y$ it is easy to show:

$$(e^{-i\pi J_y/\hbar})_{m',m} = (-1)^{j-m}\delta_{m',-m}. \tag{10}$$

First note that:

$$J_z e^{-i\pi J_y/\hbar}|m\rangle = -e^{-i\pi J_y/\hbar} J_z|m\rangle = -m\hbar e^{-i\pi J_y/\hbar}|m\rangle \tag{11}$$

which shows that $e^{-i\pi J_y/\hbar}$ changes m to $-m$ as stated in Eq. 10. Next relate the matrix element given in Eq. 10 for m to a similar one with m replaced by $m-1$ (assuming $m > -j$):

$$\begin{aligned}
\langle -m|e^{-i\pi J_y/\hbar}|m\rangle &= \frac{\langle -m|e^{-i\pi J_y/\hbar}(J_x + iJ_y)|m-1\rangle}{\sqrt{(j-m+1)(j+m)}} && (12)\\
&= \frac{\langle -m|(-J_x + iJ_y)e^{-i\pi J_y/\hbar}|m-1\rangle}{\sqrt{(j-m+1)(j+m)}} && (13)\\
&= -\frac{\Big((J_x + iJ_y)|-m\rangle,\ e^{-i\pi J_y/\hbar}|m-1\rangle\Big)}{\sqrt{(j-m+1)(j+m)}} && (14)\\
&= -\langle -m+1|e^{-i\pi J_y/\hbar}|m-1\rangle, && (15)
\end{aligned}$$

which establishes the presence of the factor $(-1)^{-m}$ in Eq. 10. Finally, the overall constant phase $(-1)^j$ can be deduced by applying the operator $\exp(-i\pi J_y/\hbar)$ to an example of the state $|j,+j\rangle$ constructed from the product of $2j$ spin-1/2 states, each with $J_z = +\hbar/2$ to show that $e^{-i\pi J_y/\hbar}|j\rangle = +|-j\rangle$.

We can now substitute Eq. 9 into Eq. 8 to replace $\mathcal{D}^j_{m',m}(\hat{n},\theta)^*$ by $(-1)^{2j-m-m'}\mathcal{D}^j_{-m',-m}(\hat{n},\theta)$ so that Eq. 8 becomes:

$$\begin{aligned}
(j,m|j_1,m_1;j_2,m_2) &= \sum_{m'=-j}^{j}\sum_{m_1'=-j_1}^{j_1}\sum_{m_2'=-j_2}^{j_2}(-1)^{j-m'}\mathcal{D}^j_{-m',-m}(\hat{n},\theta)(-1)^{j-m}\\
&\quad\cdot\mathcal{D}^{j_1}_{m_1',m_1}(\hat{n},\theta)\mathcal{D}^{j_2}_{m_2',m_2}(\hat{n},\theta)(j,m'|j_1,m_1';j_2,m_2'). && (16)
\end{aligned}$$

Finally the factors $(-1)^m$ and $(-1)^{m'}$ and the reversed signs of m and m', can be absorbed by defining new coefficients which then obey a symmetrical transformation law. Thus, define the Wigner 3-j symbols:

$$\begin{pmatrix} j & j_1 & j_2 \\ m & m_1 & m_2 \end{pmatrix} = \frac{(-1)^{j_1-j_2-m}}{\sqrt{2j+1}}(j,-m|j_1,m_1;j_2,m_2). \tag{17}$$

These new symbols now obey:

$$\begin{aligned} \begin{pmatrix} j & j_1 & j_2 \\ m & m_1 & m_2 \end{pmatrix} &= \sum_{m'=-j}^{j} \sum_{m_1'=-j_1}^{j_1} \sum_{m_2'=-j_2}^{j_2} \mathcal{D}^{j}_{m',m}(\hat{n},\theta)\mathcal{D}^{j_1}_{m_1',m_1}(\hat{n},\theta) \\ &\cdot \mathcal{D}^{j_2}_{m_2',m_2}(\hat{n},\theta) \begin{pmatrix} j & j_1 & j_2 \\ m' & m_1' & m_2' \end{pmatrix}. \end{aligned} \tag{18}$$

Note, we have shown that this symmetrical transformation law requires the $-m$ argument in the Clebsch-Gordan in the definition given in Eq. 17. Likewise the factor of $(-1)^{-m}$ is needed too. However, whether we write $(-1)^m$ or $(-1)^{-m}$ and the extra factor of $(-1)^{j_1-j_2}$ is at this point an arbitrary choice, made so that we follow the usual definition of the Wigner 3-j symbols. (Recall that $(-1)^m$ and $(-1)^{-m}$ are related by the factor $(-1)^{2m}$ which is the constant $+1$ or -1 depending on whether j is integral or 1/2-integral.)

The transformation properties given in Eq. 18 then imply that we can use these Wigner 3-j symbols to combine a collection of states made of the product of three representations of the rotation group into a state with total angular momentum zero. That is, the state

$$\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sum_{m_3=-j_3}^{j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} |j_1,m_1;j_2,m_2;j_3,m_3\rangle. \tag{19}$$

is not changed when acted on by a general rotation and hence must have $(\vec{J})^2 = 0$ and $J_3 = 0$. Since there is at most a single way to combine three angular momenta to form a state with $(\vec{J})^2 = 0$ and $J_3 = 0$, this implies that the Wigner 3-j symbols have a unique dependence on the three variables m_1, m_2 and m_3.

Since the transformation rules given in Eq. 18 are symmetrical with respect to the interchange of any j_a, m_a j_b, m_b pair, the uniqueness of the Wigner 3-j symbols implies that they must be similarly symmetrical:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = c(a,b,c) \begin{pmatrix} j_a & j_b & j_c \\ m_a & m_b & m_c \end{pmatrix}. \tag{20}$$

Given the normalization implied by Eq. 17, the Wigner 3-j symbols obey the condition:

$$\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sum_{m_3=-j_3}^{j_3} \begin{pmatrix} j_1 & j_2 & j_4 \\ m_1 & m_3 & m_4 \end{pmatrix}^2 = 1. \tag{21}$$

(The factor of $1/\sqrt{2j+1}$ in Eq. 17 implies that the sum over m_2 and m_3 in Eq. 21 should return 1 divided by the square of this extra factor. The final sum over m_1 then gives $(2j_1+1)/(2j_1+1) = 1$.)

Squaring and summing both sides of Eq. 20 over m_1, m_2 and m_3 then gives:

$$c(a,b,c)^2 = 1 \tag{22}$$

implying that $c(a,b,c)$ is a simple phase. In fact, with our choices this phase is

$$c(a,b,c) = (\epsilon_{abc})^{j_1+j_2+j_3}. \tag{23}$$

To establish Eq. 23 we must first adopt sign conventions for our Clebsch-Gordan coefficients. This contains two parts. The first is to determine how our Clebsch-Gordan coefficients change when we reverse the order of the two spins being coupled. Here we adopt the standard conventions:

$$(j,m|j_1,m_1,j_2,m_2) = (-1)^{j-j_1-j_2}(j,m|j_2,m_2,j_1,m_1). \tag{24}$$

For the case $j_1 \neq j_2$, the two quantities related by Eq. 24 are quite separate and this is simply a convention that can be imposed. For example, for the case $j_2 > j_1$ we could adopt any convention we wanted for $(j,m|j_1,m_1,j_2,m_2)$ and then view Eq. 24 as the definition of $(j,m|j_1,m_1,j_2,m_2)$ for the case $j_1 > j_2$.

However, when $j_1 = j_2$ this is a real condition and must be verified. This is not difficult to do if we consider the exchange operator S acting on the composite state $|j_1,m_1;j_1,m_2\rangle$:

$$S|j_1,m_1;j_1,m_2\rangle = |j_1,m_2;j_1,m_1\rangle. \tag{25}$$

For the case with largest value of $m_1+m_2 = 2j_1$, there is only a single state and it has eigenvalue $+1$ for S. Thus, $(j,m|j_1,m_1,j_2,m_2)$ must be symmetric. Since for this case $j = 2j_1$, this agrees with the factor $(-1)^{j-j_1-j_2} = 1$ given in Eq. 24. When we consider the next lowest value of $m_1+m_2 = 2j_1 - 1$ there are two states:

$$|j_1,j_1;j_1,j_1-1\rangle \quad \text{and} \quad |j_1,j_1-1;j_1,j_1\rangle. \tag{26}$$

From these we can form two eigenstates of S with eigenvalues ± 1. Since the lowered $|2j_1, 2j_1 - 1; j_1, j_1\rangle$ necessarily has $S = +1$ (since the lowering operation commutes with S), the new orthogonal state $|2j_1 - 1, 2j_1 - 1; j_1, j_1\rangle$ must necessarily have $S = -1$. This implies that $(2j_1 - 1, m|j_1, m_1, j_1, m_2)$ must be anti-symmetric under the exchange of m_1 and m_2, again consistent with Eq. 24. When we lower again, we must deal with three states:

$$|j_1, j_1; j_1, j_1 - 2\rangle, \quad |j_1, j_1; j_1 - 1, j_1 - 1\rangle \quad \text{and} \quad |j_1, j_1 - 2; j_1, j_1\rangle. \tag{27}$$

These can be written as two states that have $S = +1$ and one state with $S = -1$. The lowered versions of the two states we have already constructed have $S = \pm 1$. Thus, the new state with $j = 2j_1 - 2$ must have $S = +1$. This implies a symmetric Clebsch-Gordan coefficient $(2j_1 - 2, m_1 + m_2|j_1, m_1, j_1, m_2)$, again consistent with Eq. 24. It should now be clear how this pattern will continue, establishing the condition Eq. 24 for arbitrary $0 \le j \le 2j_1$.

The combination of Eq. 24 and the factor $(-1)^{j_2 - j_3}$ in the definition, Eq. 17, then establishes Eq. 23 for the case that the second and third columns are being exchanged: $a = 1$, $b = 3$, $c = 2$. The factor $(-1)^{j_2 - j_3}$ will introduce a phase $(-1)^{2(j_2 - j_3)}$ under such an interchange. This combines with the added factor $(-1)^{j_1 - j_2 - j_3}$ (coming from Eq. 24 with the appropriate change of variables) to give:

$$(-1)^{2(j_2 - j_3)}(-1)^{j_1 - j_2 - j_3} = (-1)^{j_1 + j_2 - 3j_3} = (-1)^{j_1 + j_2 + j_3} \tag{28}$$

since $4j_3$ is necessarily an even integer.

The second case we must consider is the exchange of the first and second columns, $1 \leftrightarrow 2$ in Eq. 20 or $a = 2$, $b = 1$. For this operation to be determined we must impose a further sign convention on our Clebsch-Gordan coefficients:

$$(j, j_1 - j_2|j_1, j_1; j_2, -j_2) > 0. \tag{29}$$

We will determine the phase $c(2, 1, 3)$ by comparing the sign of two quantities related by this phase:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ j_3 - j_2 & j_2 & -j_3 \end{pmatrix} = c(2, 1, 3) \begin{pmatrix} j_2 & j_1 & j_3 \\ j_2 & j_3 - j_2 & -j_3 \end{pmatrix}. \tag{30}$$

From the condition in Eq. 29 and the definition of the Wigner 3-j symbols in Eq. 17 we learn that the sign of the left-hand-side of Eq. 30 is

$$(-1)^{j_2 - j_3 - (j_3 - j_2)} = (-1)^{2(j_2 - j_3)}. \tag{31}$$

The sign of the Wigner 3-j symbol appearing on the right hand side of Eq. 30 is the sign of the quantity:

$$(-1)^{j_1-j_3-j_2}(j_2,-j_2|j_1,j_3-j_2;j_3,-j_3). \tag{32}$$

The conventions imposed in Eq. 29 imply

$$(j_2,j_1-j_3|j_1,j_1;j_3,-j_3)>0. \tag{33}$$

We can then use the lowering operator $j_1+j_2-j_3>0$ times to equate the positive sign of this quantity with the sign of the Clebsch-Gordan coefficient appearing in Eq. 32. Thus, equating the signs on the left and right hand sides of Eq. 30 we find:

$$(-1)^{2(j_2-j_3)}=c(2,1,3)(-1)^{j_1-j_3-j_2} \tag{34}$$

or $c(2,1,3)=(-1)^{-j_1+3j_2-j_3}=(-1)^{j_1+j_2+j_3}$ as claimed.