Physics GR6038 Professor Christ
due 2/02/2025

Problem Set #12

Don't miss the second problem on the next page.

42. The nucleus of a deuterium atom has an electric quadrupole moment described by the operator:

$$Q_{i,j} = Q\left(I_i I_j + I_j I_i - \frac{4}{3}\hbar^2 \delta_{i,j}\right)$$

where $\vec{I}$ is the angular momentum operator of the spin-1 nucleus and Q is a constant. This quadrupole moment produces an electrostatic potential:

$$V(\vec{r}) = \sum_{ij}(\hat{r}_i \hat{r}_j - \frac{1}{3}\delta_{i,j})Q_{i,j}\frac{1}{r^3}.$$

You should ignore the effects of the nuclear magnetic moment and Lamb shift but include the effects of fine structure when answering (a), (b) and (c) below.

(a) Describe the degeneracies present among the deuterium energy eigenstates and deduce which states should be used for a first-order, perturbative treatment of the effects of $V(\vec{r})$ on the atom which deals properly with these degeneracies.

(b) Show that a symmetric, traceless second rank tensor has five independent components and conclude that Q_{ij} forms an invarinat tensor with $j = 2$.

(c) Using the states chosen in (a), compute the energy levels of the deuterium atom to first order in Q as follows:

i. Use the Wigner-Eckart theorem to show that

$$\langle j, m|\hat{r}_i \hat{r}_j - \frac{1}{3}\delta_{i,j}|j, m'\rangle = c\langle j, m|J_i J_j + J_j J_i - \frac{2}{3}j(j+1)\hbar^2\delta_{i,j}|j, m'\rangle.$$

Here $|j, m\rangle$ are the usual Hydrogen atom fine structure eigenstates with total orbital plus spin angular momentum quantum numbers $\vec{J}^2 = \hbar^2 j(j+1)$ and $J_z = \hbar m$.

ii. To determine c note that if the two second-rank tensor operators above are called O^r_{ij} and O^J_{ij}, then c can be determined by comparing the matrix elements of $J_i O^r_{ij} J_j$ and $J_i O^J_{ij} J_j$. Here i and j are summed from 1 to 3.

iii. The combination $J_i O^J_{ij} J_j$ is not difficult to evaluate. While $J_i O^r_{ij} J_j$ looks more difficult, it simpfies quickly when written down and the properties of the Pauli matrices exploited.

iv. You may use the relation:

$$\langle n, l|\frac{1}{r^3}|n, l\rangle = \frac{1}{a_0^3 n^3 l(l+1/2)(l+1)}.$$

(d) Make a rough numerical comparison of the splitting obtained in (b) and the usual hyperfine separation.

43. Consider a hydrogen atom placed in a weak uniform magnetic field $\vec{B}$. The Hamiltonian can be written:

$$H = H_0 + H_{\mathrm{FS}} + H_{\mathrm{B}}$$

where

$$\begin{aligned} H_0 &= \frac{\vec{p}^2}{2m} - \frac{e^2}{|\vec{r}|} \\ H_{\mathrm{FS}} &= -\frac{|\vec{p}|^4}{8m^3c^2} + \frac{e^2}{4m^2c^2}\frac{(\vec{L}\cdot\vec{S})}{|\vec{r}|^3} \\ H_{\mathrm{B}} &= \frac{e}{2mc}\vec{B}\cdot(\vec{L}+2\vec{S}) \end{aligned}$$

Work with eigenstates of $H_0 + H_{\mathrm{B}}$. Find the resulting spectrum of H treating that the matrix elements of H_{FS} as small compared to those of H_0 but including the case that the matrix elements of H_{FS} and H_{B} are of the same size. This problem asks that you generalize the two cases considered in class where the magnetic interaction is small compared to the spin-orbit coupling (Zeeman effect) or large compared to the spin-orbit coupling (Pashen-Back effect) to the case where these two small effects are of the same size.