Physics GR6038 Professor Christ
due 02/9/2025

Problem Set #13

Don't miss the second problem on the next page.

44. Return to problem 27 in problem set #8 from last semester and express the time-dependent Schrödinger equation is terms of the applied frequency, here called ω, and the Larmor frequencies $\omega_0 = \gamma B_0$ and $\omega_1 = \gamma B_1$ at which the electron would precess if only the original field $B_0\hat{z}$, or a time-independent version of the rotating field B_1, were present:

$$i\frac{d}{dt}|\psi(t)\rangle = -\Big\{\frac{\omega_0}{2}\sigma_3 + \frac{\omega_1}{2}\Big(\sigma_1\cos(\omega t) - \sigma_2\sin(\omega t)\Big)\Big\}|\psi(t)\rangle.$$

(a) Use Python to evaluate the analytic solution you obtained when working problem 27. You should choose time units so that $\omega_0 = 2\pi$, making the period of Larmor precession equal to 1 in those units. Use the initial state $(|+\frac{1}{2}\rangle + |-\frac{1}{2}\rangle)/\sqrt{2}$, $\omega_1 = \omega_0/5$ and plot the expectation value for the three components of the electron's spin as a function of time t for $0 \le t \le 5$. Plot the graph using 5,000 values for the time t separated by a time interval $\Delta t = 0.001$ for the value $\omega = 2$. (Only this final plot and the second one requested below need to be included when you submit your solution.)

(b) Show that the time-dependent Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle.$$

is also solved by the $\Delta t \to 0$ limit of the product

$$|\psi(t)\rangle = \prod_{n=0}^{N-1} e^{-iH(t_n)\Delta t/\hbar}|\psi(0)\rangle.$$

where $N = t/\Delta t$ and $t_n = n\Delta t$ and the factors in the product are ordered so that those at later time appear farther to the left. With what power of Δt do the errors in $|\psi(t)\rangle$ in the above product vanish as $\Delta t \to 0$?

(c) Can you find a variant of this method with smaller errors that can be evaluated with the same or less computational cost? (You might take inspiration from the leap-frog integration method introduced in the Schrodinger.ipynb example posted on the web.)

(d) Use the method in part b) to obtain an alternative numerical solution to the specific problem posed in part a). Include with your problem solutions a graph of this numerical solution using the same time step $\Delta t = 0.001$ and time interval $0 \le t \le 5$ as used in part a).

[It may be helpful to begin with a working sample Jupyter notebook larmor.ipynb, posted on the course web site, which addresses simple Larmor precession and then to modify it to solve this homework problem.]

45. Three spin-1 particles with angular momentum operators $\vec{S}_1$, $\vec{S}_2$ and $\vec{S}_3$ are located at fixed positions . These three spins interact through the Hamiltonian

$$H = \lambda \left(\vec{S}_1 \cdot \vec{S}_2 + \vec{S}_1 \cdot \vec{S}_3 + \vec{S}_2 \cdot \vec{S}_3 \right)$$

(a) Find the allowed energies of this systems and their multiplicities. Make a simple table showing each energy and its multiplicity.

(b) Assume that each of these three spins has a magnetic dipole moment $\vec{\mu}^{(i)} = \gamma \vec{S}^{(i)}$. Apply a magnetic field so that the magnetic energy:

$$H_{\mathrm{B}} = -\gamma \vec{B} \cdot \left(\vec{S}_1 + \vec{S}_2 + \vec{S}_3 \right),$$

is added to the Hamiltonian. What are the allowed energies and their multiplicities?

(c) Finally consider the case that the gyromanetic ratio γ_1 for particle 1 is different from that of particles 2 and 3 so that

$$H_{\mathrm{B}} = -\vec{B} \cdot \left(\gamma_1 \vec{S}_1 + \gamma_{23} (\vec{S}_2 + \vec{S}_3) \right),$$

Find the resulting spectrum of $H + H_{\mathrm{B}}$ in the limits of weak and strong $\vec{B}$ and describe the corresponding eigenstates. For the weak field case work to first order in $\vec{B}$ and for the strong field case work to first order in λ.