

Problem Set #18

Don't miss problem 63 on the next page

61. N spinless bosons of mass m move in the three-dimensional harmonic oscillator potential

$$V(r) = \frac{1}{2}m\omega^2\vec{r}^2.$$

Find the energies and degeneracies of the three lowest energy levels.

62. Consider a system of identical bosons confined to a volume V . Let $\{\phi_k(\vec{r})\}$ be a complete, orthonormal set of single-particle wave functions. If $a_k^\dagger$ is the creation operator for the state ϕ_k in our second-quantized formalism, define the operator

$$\Psi(\vec{r}) = \sum_k a_k \phi_k(\vec{r}).$$

- (a) Show that when applied to the vacuum, the operator

$$\int_V f(\vec{r}) \Psi^\dagger(\vec{r}) d^3r$$

creates a single particle state with wave function $f(\vec{r})$.

- (b) Consider an operator $\mathcal{V}$ which when acting on N -particle states has the form

$$V_N = \frac{1}{2} \sum_{j=1}^N \sum_{\substack{i=1 \\ i \neq j}}^N V(\vec{r}_i, \vec{r}_j)$$

where $\vec{r}_1 \dots \vec{r}_N$ are the coordinates operators for the N particles and $V(\vec{r}, \vec{r}')$ is a real function of the variables $\vec{r}$ and $\vec{r}'$. Using the second quantized formalism, $\mathcal{V}$ can be written:

$$\mathcal{V} = \frac{1}{2} \int d^3r \int d^3r' \Psi^\dagger(\vec{r}') \Psi^\dagger(\vec{r}) V(\vec{r}, \vec{r}') \Psi(\vec{r}') \Psi(\vec{r}).$$

Show that this formula is correct for matrix elements of the form:

$$\langle n'_1, n'_2, \dots, n'_k \dots | \mathcal{V} | n_1, n_2, \dots, n_k \dots \rangle$$

for the following two cases:

- i. When $n'_k = n_k$ for all states k .
- ii. When $n'_k = n_k$ for all states k except for the four distinct states: $k = a, b, c, d$ and for these cases:

$$n'_a = n_a + 1, \quad n'_b = n_b + 1, \quad n'_c = n_c - 1, \quad n'_d = n_d - 1$$

- (c) If the term $i = j$ is included in the definition of V_N what is the corresponding formula for $\mathcal{V}$?

63. A system of N spinless bosons in three dimensions is confined inside a cube of side L by the potential

$$V(x, y, z) = \begin{cases} 0 & \text{if } 0 \leq x \leq L, 0 \leq y \leq L, 0 \leq z \leq L \\ \infty & \text{otherwise.} \end{cases}$$

- (a) If each pair of bosons interacts through the potential

$$V(\vec{r}, \vec{r}') = \lambda \delta^3(\vec{r} - \vec{r}'),$$

use the expression given in Problem 62. b) to write down the Hamiltonian for the system using the “second quantized” field operator $\Psi(\vec{r})$ and its hermitian conjugate $\Psi(\vec{r})^\dagger$ which destroy and create a particle at the position $\vec{r}$.

- (b) What is the ground state and ground state energy to 0^{th} order in λ for the Hamiltonian of (a)?
- (c) What is the shift in the ground state energy to first order in λ ?