Physics GR6038 Professor Christ
due 04/06/2025

Problem Set #19

64. Consider a system of N, non-interacting, spin-1/2 fermions. If $\{\phi_k(\vec{r}, m_s)\}$ is a complete set of single particle energy eigenstates with energies $\{E_k\}$ and a_k is the usual destruction operator for the state ϕ_k, show that the "second quantized" operator

$$\Psi(\vec{r}, m_s, t) = \sum_k a_k \phi_k(\vec{r}, m_s) e^{-iE_k t/\hbar}$$

obeys

$$i\hbar \frac{\partial}{\partial t} \Psi(\vec{r}, m_s, t) = -\Big[H, \Psi(\vec{r}, m_s, t)\Big]$$

where H is the complete Hamiltonian for the N-particle system. (Note, that this equation involves a commutator even though the operators a_k and $a_k^\dagger$ obey anti-commutation relations.) If the single particle Hamiltonian is $\vec{p}^{\,2}/2m + V(\vec{r})$, show that Ψ obeys

$$i\hbar \frac{\partial}{\partial t} \Psi(\vec{r}, m_s, t) = \Big[-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{r})\Big] \Psi(\vec{r}, m_s, t).$$

65. (Baym, Chapter 19, Problem 2)

    (a) Find a general formula, accurate to first order in the inter-particle interaction $V(\vec{r} - \vec{r}\,')$ for the energy of an $N+1$ particle system of spin-1/2 fermions with one particle of momentum $\vec{p}$ outside of an N-particle Fermi sea. Your answer should be expressed as the difference of the energy of the specified state and the energy of the N-particle ground state. (This might be most easily solved by using creation and annihilation operators to express the difference as the matrix element of a difference of operators—a difference which can be simplified using anti-commutation relations.)

    (b) Evaluate your result explicitly for the case of a Coulomb interaction (include the effects of a uniform positive charge density making the whole system neutral).

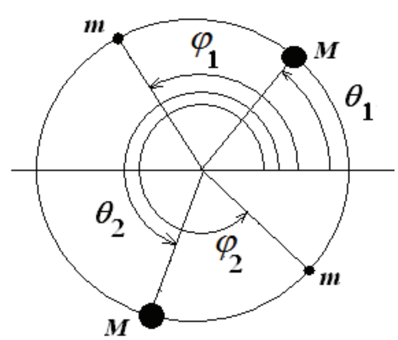
66. Consider a system of N identical spin-1/2 particles moving independently in a three-dimensional harmonic potential. The total Hamiltonian is:

$$H = \sum_{i=1}^{N} \left(\frac{\vec{p}_i^{\,2}}{2m} + \frac{1}{2}\omega^2 \vec{r}_i^{\,2} \right).$$

(a) Find (but do not solve) an equation relating N and the energy of the ground state and determine the degeneracy of the ground state.

(b) If the particles have magnetic moment $\vec{\mu} = \gamma \vec{s}$ and the system is placed in a uniform magnetic field $\vec{B}$, find the energy of the ground state, $E_{\rm gnd}$ as a function of B. Draw rough graphs of your result for $E_{\rm gnd}$ for fixed, weak B as a function of N and for fixed N as a function of B. Be careful that as B increases you are always dealing with the lowest energy state. (You need only give a specific answer when $|\vec{B}| < 2\omega/\gamma$.)

67. Two particles of mass m and two particles of mass M move on a circle of radius R. Locate the particles with mass m with the angles, φ_1 and φ_2 and those with mass M with θ_1 and θ_2. The particles interact as hard spheres of radius $r/2$ so the interaction energy between the particles at the angles ϕ_i and θ_j is:



$$V(\theta_i - \theta_j) = \begin{cases} 0 & |\varphi_i - \theta_j| > r/R \\ \infty & |\varphi_i - \theta_j| < r/R. \end{cases}$$

Assume that the particles are arranged as shown in the figure, that $M \gg m$ and neglect r/R.

(a) If the two heavy particles are at fixed locations θ_1 and θ_2, find the ground state energy for the light particles.

(b) Use the Born-Oppenheimer approximation to find the ground state and ground state energy for the entire system.