Physics GR6038 Professor Christ
due 05/04/2025

Problem Set #23

Don't miss the problems on the next page

Improvements made to the wording of problems 79 and 81

78. Generalize the analysis given in class of the resonance-decay problem to the case where the decaying state has a general integer angular momentum l:

- Express $H = H_0 + V$.
- Assume H_0 has two types of eigenstates: $\{|m\rangle\}_{-l \le m \le l}$ and $|\phi_{\vec{p}}\rangle$ with:

$$H_0|m\rangle = E_d|m\rangle \qquad H_0|\phi_{\vec{p}}\rangle = \frac{p^2}{2M}|\phi_{\vec{p}}\rangle,$$

where $|m\rangle$ and $|\phi_{\vec{p}}\rangle$ obey the normalization conditions:

$$\langle m'|m\rangle = \delta_{m',m} \quad \langle\phi_{\vec{p}'}|\phi_{\vec{p}}\rangle = (2\pi\hbar)^3\delta(\vec{p}' - \vec{p}).$$

- Assume that the only non-zero matrix elements V are

$$V^*_{m,\vec{p}} = V_{\vec{p},m} = \langle\phi_{\vec{p}}|V|m\rangle \equiv F(p)Y_{l,m}(\hat{p}).$$

(a) Find the incoming states $|\psi^{in}_{\vec{p}}\rangle$ in terms of $|m\rangle$ and $|\phi_{\vec{p}}\rangle$.

(b) Calculate the scattering amplitude $f(\theta, \phi)$.

(c) Find the scattering phase shift $\delta_l(p)$.

79. Using stationary state scattering theory, consider scattering from a hard sphere of radius a. In contrast with the low energy limit discussed in class to which only the s-wave phase shift contributes, explore the high-energy limit in which a sum over the contributions of many partial waves must be evaluated. You might do this using a simple Python program. Examine the particular case where the input momentum is $400\hbar/a$. What is your result for the total cross section at this incident momentum and determine how many terms are needed in the partial wave sum to obtain a result with a accuracy less than 1%?

80. Find the average electric field at the point $\vec{r}$ and time t for each of the following initial states described at $t = 0$:

(a) $a_i(\vec{k}_{\vec{j}})^\dagger|0\rangle$

(b) $\frac{1}{\sqrt{1+|b|^2}}\{|0\rangle + b\, a_i(\vec{k}_{\vec{j}})^\dagger|0\rangle\}$

(c) $\sum_{m=0}^{\infty} \frac{b^m}{m!}\left\{a_i(\vec{k}_{\vec{j}})^\dagger\right\}^m|0\rangle e^{-|b^2|/2}$.

81. Consider an electromagnetic field obeying periodic boundary conditions on the walls of a cube of volume V, described by the vector potential $\vec{A}(\vec{r},t)$ satisfying the Coulomb gauge condition $\vec{\nabla}\cdot\vec{A}=0$. Given:

$$\begin{aligned}
\vec{A}(\vec{r}) &= \sum_{j,n}\mathcal{E}_j(\vec{k}_n)\Big(q_j^{re}(\vec{k}_n)+iq_j^{im}(\vec{k}_n)\Big)\frac{e^{i\vec{k}\cdot\vec{r}}}{\sqrt{V}} \\
p_j^{re}(\vec{k}_n) &= \frac{1}{2\pi c^2}\frac{d}{dt}q_j^{re}(\vec{k}_n), \quad p_j^{im}(\vec{k}_n)=\frac{1}{2\pi c^2}\frac{d}{dt}q_j^{im}(\vec{k}_n) \qquad (1)\\
a_j(\vec{k}_n) &= \frac{1}{2}\left\{\sqrt{\frac{k_n}{2\pi\hbar c}}\Big(q_j^{re}(\vec{k}_n)+iq_j^{im}(\vec{k}_n)\Big)+i\sqrt{\frac{2\pi c}{\hbar k_n}}\Big(p_j^{re}(\vec{k}_n)+ip_j^{im}(\vec{k}_n)\Big)\right\},
\end{aligned}$$

derive the right-most equality in each of the formulae:

$$\begin{aligned}
\vec{B}(\vec{r}) &= i\sum_{j,n}\sqrt{\frac{2\pi\hbar c}{Vk_n}}\,\vec{k}_n\times\hat{\mathcal{E}}_j(\vec{k}_n)\Big\{a_j(\vec{k}_n)e^{i\vec{k}_n\cdot\vec{r}}-a_j(\vec{k}_n)^\dagger e^{-i\vec{k}_n\cdot\vec{r}}\Big\} \\
\vec{E}(\vec{r}) &= i\sum_{j,n}\sqrt{\frac{2\pi\hbar c}{Vk_n}}\,|\vec{k}_n|\hat{\mathcal{E}}_j(\vec{k}_n)\Big\{a_j(\vec{k}_n)e^{i\vec{k}_n\cdot\vec{r}}-a_j(\vec{k}_n)^\dagger e^{-i\vec{k}_n\cdot\vec{r}}\Big\} \\
H &= \frac{1}{8\pi}\int d^3r\Big\{\vec{E}^2(\vec{r})+\vec{B}^2(\vec{r})\Big\}=\sum_{j,n}\hbar\omega\Big\{a_j(\vec{k}_n)^\dagger a_j(\vec{k}_n)+\frac{1}{2}\Big\} \\
\vec{P} &= \frac{1}{4\pi c}\int d^3r\ \vec{E}(\vec{r})\times\vec{B}(\vec{r})=\sum_{j,n}\hbar\vec{k}_n a_j(\vec{k}_n)^\dagger a_j(\vec{k}_n).
\end{aligned}$$

All of the equations above can be considered as time-independent operator equations with the exception of Eqs. (1) which should be viewed as classical equations showing how the $\dot{q}_i^{re}(\vec{k}_n)$ and $\dot{q}_i^{im}(\vec{k}_n)$ appearing in $\vec{E}$ can be replaced by the canonical momenta variables $p_i^{re}(\vec{k}_n)$ and $p_i^{im}(\vec{k}_n)$.

82. (**This is an extra problem which will not be included in your homework grade.**) Two classical magnetic moments $\vec{\mu}_1$ located at $\vec{r}_1$ and $\vec{\mu}_2$ located at $\vec{r}_2$ interact with the quantum electromagnetic field through the Hamiltonian:

$$H=-\vec{\mu}_1\cdot\vec{B}(\vec{r}_1)-\vec{\mu}_2\cdot\vec{B}(\vec{r}_2)$$

(a) Express the magnetic field $\vec{B}(\vec{r})$ above in terms of the creation and annihilation operators $a_i(k_{\vec{l}})$ and $a_i^\dagger(k_{\vec{l}})$ for photons with wave number $\vec{k}_l$ and polarization $\hat{\mathcal{E}}_i(\vec{k}_l)$ in a finite volume of side L.

(b) Using your expression from part (a), write down the complete Hamiltonian for the electromagnetic degrees of freedom in terms of the creation and annihilation operators.

(c) Find the energy eigenvalues and eigenstates of the resulting Hamiltonian. In particular, find the dependence of the resulting ground state energy on $\vec{\mu}_1$, $\vec{\mu}_2$ and $\vec{r}_2-\vec{r}_1$ in the limit $L\to\infty$. [Hint: This can be done by using new annihilation and creation operators obtained by shifting $a_i(\vec{k}_l)$ and $a_i^\dagger(\vec{k}_l)$ by complex constants].

[In Coulomb gauge all magnetic forces must come from the exchange of quantum photons and this problem shows how this can work.]