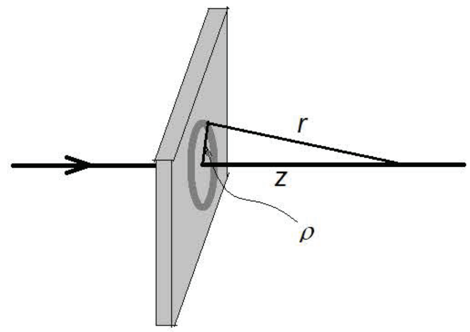
Quantum mechanical propagation through a dilute medium using stationary state scattering theory

Consider a plane wave $e^{ip_z z/\hbar}$ passing through a thin slab of thickness Δz of material made up of a dilute distribution of scattering centers of density n. Refer to the drawing and choose the z-direction to point toward the right. We would like to determine the effect on the incident wave of passing through the slab by using a stationary-state scattering description of the wave function on the right side of the slab.



We can calculate the wave function at a distance z from the slab by adding the scattered wave coming from all of the scattering centers in a ring of radius ρ and then integrating over all such rings:

$$\psi(z) = e^{ip_z z/\hbar} + n\Delta z 2\pi \int_0^{\rho_{\max}} \rho d\rho f(\theta) \frac{e^{ipr}}{r}, \tag{1}$$

where for simplicity we have assumed that the scattering amplitude $f(\theta, \phi)$ does not depend on ϕ. We have also introduced an upper limit $\rho_{\max}$. Since we are considering elastic scattering, where the scattered wave has the same energy as the incident wave we must require that the target particle cannot recoil. If the target particle is viewed as held in place by a potential created by its neighboring particles in the slab, the elastic amplitude will go rapidly to zero when the scattering angle becomes so large that it becomes unlikely that the target particle will remain in its ground state.

Next we change variables from ρ to r. Since $r^2 = z^2 + \rho^2$, for fixed z $\rho d\rho = rdr$ so that Eq. (1) becomes:

$$\begin{aligned}
\psi(z) &= e^{ip_z z/\hbar} + n\Delta z 2\pi \int_0^{r_{\max}} dr f\Big(\arccos(z/r)\Big) e^{ipz\xi/\hbar} && (2)\\
&= e^{ip_z z/\hbar} + n\Delta z 2\pi \int_0^{\xi_{\max}} d\xi f\Big(\arccos(1/\xi)\Big) \frac{\hbar}{ip}\frac{d}{d\xi} e^{ipz\xi/\hbar} && (3)
\end{aligned}$$

where we have introduced $\xi = r/z$. Integrating by parts we find:

$$\begin{aligned}
\psi(z) &= e^{ip_z z/\hbar} + \frac{n\Delta z 2\pi\hbar}{ip} f\Big(\arccos(1/\xi)\Big) e^{ipz\xi/\hbar}\Bigg|_1^{\xi\max} && (4)\\
&\qquad - \frac{n\Delta z 2\pi\hbar}{ip} \int_0^{\xi\max} d\xi \frac{d}{d\xi}\Big\{ f\Big(\arccos(1/\xi)\Big)\Big\} e^{ipz\xi/\hbar}
\end{aligned}$$

This result can be simplified by recognizing that the middle term in Eq. (4) comes only from the $\xi = 1$ case because $\xi_{\max}$ is chosen sufficiently large that

$$f\Big(\arccos(1/\xi_{\max})\Big) = 0. \tag{5}$$

In addition, the right-most term on the right-hand side of Eq. (4) has the same form as that in Eq. (2) so that a second integration by parts can be performed introducing a further factor of $1/z$ which will vanish for large z.

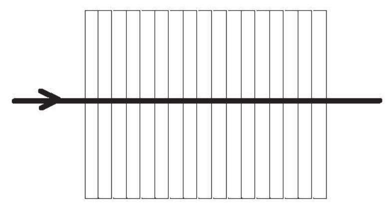
We find

$$\psi(z) = e^{ip_z z/\hbar} - \frac{n\Delta z 2\pi\hbar}{ip} f(0) e^{ipz/\hbar} \tag{6}$$

which implies that:

$$\begin{aligned} \left|\psi(z)\right|^2 &= \left|1 - \frac{n\Delta z 2\pi\hbar}{ip} f(0)\right|^2 && (7) \\ &= 1 - n\Delta z \frac{4\pi\hbar}{p} \mathrm{Im} f(0) && (8) \\ &= 1 - n\Delta z\, \sigma_{\mathrm{tot}} && (9) \end{aligned}$$

which can be viewed as a consistency check or a derivation of the optical theorem since we find that the probability flux has been reduced by precisely the factor expected if it is recognized that the probability scattered flux scattered out of the beam is determined by the total crossection σ as in Eq (9).



Finally consider the scattering of an incident wave $e^{ipz/\hbar}$ by a slab of thickness W as suggested by the collection of N slabs of thickness $\Delta z = W/N$ shown in the figure. Write the scattered wave $\psi(z)$ where where $z > W$ in form:

$$\psi(z) = T(W) e^{ipz/\hbar}. \tag{10}$$

Equation (6) implies that if we consider adding an $N + 1^{st}$ slab we can show that the coefficient $T(W)$ must obey the equation:

$$\frac{d}{dW} T(W) = \frac{i2\pi n\hbar}{p} f(0)\, T(W). \tag{11}$$

The discussion above considers the change to the scattered wave seen far from the scattering slabs suggesting it may not apply to a slab added adjacent to the right side of the combined slab. However, this assumed form will be correct for the wave passing though most of the N thin slabs which are far from the right edge since they out-number the few slabs which are near the $N + 1^{st}$ slab that has been added.

Since $T(0) = 1.$ we can conclude that:

$$T(W) = e^{i\frac{2\pi n\hbar}{p} f(0) W} \tag{12}$$

Thus, after passing through the slab of thickness W the phase of the wave has changed by

$$e^{i\frac{pW}{\hbar}\left(1 + \frac{2\pi n f(0)\hbar^2}{p^2}\right)}. \tag{13}$$

We can view the slab as introducing an index of refraction $n_{\mathrm{refraction}} = 1 + \frac{2\pi n f(0)}{p^2}$.