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Fermi's golden rule without and with momentum conservation

Here we give the derivation of Fermi's golden rule for the decay of an unstable state derived in first-order perturbation theory that was presented in class. This note is divided into two sections. The first is the convention case in which the unperturbed Hamiltonian, H_0, whose eigenstates include the unstable state and its decay products, contains a potential energy function at a fixed location so momentum is not conserved. In the second section we assume that H_0 is invariant under translations and any potential energy function that appears depends on coordinate differences and is therefore translationally invariant so momentum is conserved.

1 Decay of an unstable state without momentum conservation

We begin with a Hamiltonian $H = H_0 + V(t)$ where the eigenstates of H_0 are a discrete state $|d\rangle$ with H_0 eigenvalue E_d and a set of non-normalizable, infinite-volume states $|\phi(\vec{p})\rangle$ obeying:

$$\begin{aligned} H_0|d\rangle &= E_d|d\rangle \qquad H_0|\phi(\vec{p})\rangle = E_p|\phi(\vec{p})\rangle && (1) \\ \langle d|d\rangle &= 1 \qquad \langle\phi(\vec{p}\,')|\phi(\vec{p})\rangle = \delta^3(\vec{p}\,' - \vec{p}) && (2) \end{aligned}$$

Here $|d\rangle$ will become unstable when a small perturbation $V(t)$ is added to H_0, defining a new Hamiltonian $H = H_0 + V(t)$ which couples the state $|d\rangle$ to its decay products $|\phi(\vec{p})\rangle$. The infinite-volume states $|\phi(\vec{p})\rangle$ are required for unstable particle decay. If the decay products were unable to escape to infinity but instead bounced back after reaching the walls of a finite volume, the resulting finite-volume energy eigenstates would have precise, discrete energies and no decay phenomena would occur.

In most applications the potential V coupling the state $|d\rangle$ to its decay products is time independent and this time independence is required for the validity of Fermi's golden rule. However, in the special case that the time dependence of $V(t)$ comes from only a few frequencies $\{\omega_n\}_{1\leq n\leq N}$ then Fermi's golden rule continues to be useful with the energies $\{\hbar\omega_n\}_{1\leq n\leq N}$ simply shifting the argument of the decaying state energy in Fermi's golden rule.

The rate for the decay of the state $|d\rangle$ can be computed using time-dependent perturbation theory, starting with the interaction picture formula for the time-dependence of the state:

$$\begin{aligned} |\psi(t)\rangle_I &= a(t)|d\rangle + \int d^3 b_{\vec{p}}(t)|\phi(\vec{p})\rangle \qquad (3) \\ &= T\left\{e^{-\frac{i}{\hbar}\int_0^t V_I(t')}\right\}|d\rangle. \qquad (4) \end{aligned}$$

The probabililty $P(t)$ that the state has decayed can be computed from the integrated probability that the decay product states $|\phi(\vec{p})\rangle$ are present:

$$P(t) = \int d^3p\, |b_{\vec{p}}(t)|^2 \qquad (5)$$

while the coefficients $b_{\vec{p}}(t)$ can be determined to first-order in perturabtion theory from Eq. (4):

$$\begin{aligned} b_{\vec{p}}(t) &= \int_0^t \langle\phi(\vec{p})|V_I(t)|d\rangle dt' \qquad (6) \\ &= \int_0^t \langle\phi(\vec{p})|V(t)|d\rangle e^{-\frac{i}{\hbar}(E_d-E_p)t'} dt' \qquad (7) \\ &= \langle\phi(\vec{p})|V(t)|d\rangle \frac{e^{-\frac{i}{\hbar}(E_d-E_p)t}-1}{E_d-E_p}, \qquad (8) \end{aligned}$$

where in Eq. (7) we have used the definition of a general interpicture operator:

$$O_I(t) = e^{\frac{i}{\hbar}\hbar H_0 t} O\, e^{-\frac{i}{\hbar}\hbar H_0 t}. \qquad (9)$$

We can determine $P(t)$ by substituting Eq. (8) into Eq. (5):

$$P(t) = \int d^3p |V_{\vec{p},d}|^2 \frac{4\sin^2\left(\frac{1}{\hbar}(E_d-E_p)t\right)}{(E_d-E_p)^2}, \qquad (10)$$

where we have abreviated our expression for $\langle\phi(\vec{p})|V(t)|d\rangle = V_{\vec{p},d}$. While not yet a decay rate this result has physical importance. We can view the integrand as providing the momentum distribution of the decay products as a function of the time elapsed since $t = 0$ when the state was precisely $|d\rangle$. This equation shows a function of energy E_p which is peaked at $E_p = E_d$ but with a width or spread of energies equal to $\hbar/t$ and a height using L'hopital's rule growing as t^2. Thus, one expects the probability of decay to grow proportional to t where the coefficient of proportionalityis the desired decay rate.

For large time, it is not difficult to determine this proportionality constant. In this limit the ratio $\sin^2\left(\frac{i}{\hbar}(E_d - E_p)t\right)/(E_d - E_p)^2$ in Eq. (10) becomes very sharply peaked around $E_p = E_d$ so the integral in that equation can be approximated by evaluating the other factors in the integrand at this maximal point, $E_p = E_d$ and describing the dependence of the integrand as coming entirely from the ratio $\sin^2\left(\frac{i}{\hbar}(E_d - E_p)t\right)/(E_d - E_p)^2$. In this approximation we can make the replacement:

$$\frac{\sin^2\left(\frac{1}{\hbar}(E_d - E_p)t\right)}{(E_d - E_p)^2} = c(t)\delta(E_d - E_p). \tag{11}$$

We can determine $c(t)$ in this equation by evaluating the integral:

$$\begin{aligned} c(t) &= \int_{-\infty}^{\infty} dE_p \frac{\sin^2\left(\frac{1}{\hbar}(E_d - E_p)t\right)}{(E_d - E_p)^2} && (12) \\ &= \frac{t}{\hbar}\int_{-\infty}^{\infty} dx \frac{\sin^2(x)}{x^2} && (13) \\ &= \frac{t}{\hbar}\frac{\pi}{2}. && (14) \end{aligned}$$

This result can be substituted into Eq. (10) to obtain Fermi's golden rule for the decay rate γ:

$$\gamma = \frac{2\pi}{\hbar}\int d^3p |V_{\vec{p},d}|^2 \delta(E_d - E_p), \tag{15}$$

where the coefficient of t obtained from that substitution has been identified as the decay rate.

We should emphasize that this is a first-order perturbation theory result. If we had actual exponential decay with the probability of remaining in the intial state decaying as $\exp(-\gamma t)$ then we would expect

$$P(t) = 1 - e^{-\gamma t} \approx \gamma t \tag{16}$$

if we work to lowest order in the decay rate γ. The other unphysical property of our perturbative treatment is the conclusion following Eq. (10) that the width of the energy distribution approaches zero as the time grows. As we know from our WKB treatment of α decay, the final energy distribution for very large time has a non-zero width $\gamma\hbar$, as might be guessed from the time-energy uncertainty relation.

2 Decay of an unstable state with momentum conservation

While the result given in Eq. (15) is very useful, it does not capture the important case in which sufficient elements of the problem have been explicitly included in its description that momentum is conserved. This must be a case in which both the initial decaying particle and final decay products move unconstrained in infinite volume. The simplest example of such a decay would have two particles, 1 and 2, as the decay products and generalize Eqs. (1) and (2) to:

$$\begin{aligned} H_0|d(\vec{P}\,)\rangle &= E_{d,P}|d(\vec{P}\,)\rangle && (17)\\ H_0|\phi_i(\vec{p}\,)\rangle &= E_{i,p}|\phi_I(\vec{p}\,)\rangle \text{ for } i\in\{1,2\} && (18)\\ \langle d(\vec{P}\,')|d(\vec{P}\,)\rangle &= \delta^3(\vec{P}\,'-\vec{P}\,) && (19)\\ \langle \phi_i(\vec{p}\,')|\phi_j(\vec{p}\,)\rangle &= \delta_{ij}\delta^3(\vec{p}\,'-\vec{p}\,) \text{ for } i,j\in\{1,2\}. && (20) \end{aligned}$$

Now the operator V which connects the decaying state with the decay products will be assumed to conserve momentum which requires that for our non-normalizable states the matrix elements of V must have the form:

$$[\langle\phi_1(\vec{p}_1)|\otimes\langle\phi_2(\vec{p}_2)]\,|V|d(\vec{P}\,)\rangle = V_{\vec{p}_1,\vec{p}_2,d}\,\delta^3(\vec{p}_1+\vec{p}_2-\vec{P}\,). \qquad (21)$$

To calculate a physical process we must deal with a normalized intial decaying state. We introduce the normalized momentum-space wave function $\widetilde{\psi}_d(\vec{P})$ for initial decaying state and following the approach in Section 1, write the time dependent state $|\psi(t)\rangle$ as

$$|\psi(t)\rangle = \int d^3P\, a(\vec{P},t)|d(\vec{P}\,)\rangle + \int d^3p_1 d^3p_2\, b_{\vec{p}_1,\vec{p}_2}(t)|\phi_1(\vec{p}_1)\rangle\otimes|\phi_2(\vec{p}_2)\rangle, \quad (22)$$

where we impose the initial conditions:

$$a_d(\vec{P},0) = \widetilde{\psi}_d(\vec{P}\,) \qquad b_{\vec{p}_1,\vec{p}_2}(0) = 0. \qquad (23)$$

Working to first order in V as done in Section 1 we find the probability that the decay has occured to be

$$P(t) = \int d^3p_1 d^3p_2 \, |V_{\vec{p}_1,\vec{p}_2,d}|^2 \left|\widetilde{\psi}_d(\vec{p}_1+\vec{p}_2)\right|^2 \frac{4\sin^2\left(\frac{1}{\hbar}(E_{d,P}-E_{1,p_1}-E_{1,p_2})t\right)}{(E_{d,P}-E_{1,p_1}-E_{1,p_2})^2}, \tag{24}$$

where we have used the momentum delta function in Eq. (21) to perform the integral over $\vec{P}$ that appears in the definition of the initial state.

To obtain our momentum-converving version of Fermi's golden we must take two limits. In the first we assume that the time t is sufficiently large that we can make the replacement:

$$\frac{\sin^2\left(\frac{1}{\hbar}(E_{d,P}-E_{1,p_1}-E_{1,p_2})t\right)}{(E_{d,P}-E_{1,p_1}-E_{1,p_2})^2} \approx \frac{\pi}{2}\frac{t}{\hbar}\delta(E_{d,P}-E_{1,p_1}-E_{1,p_2}). \tag{25}$$

For the second we assume that on the scale of the momenta that enter the decay, the momentum distribution of the initial state, given by the initial state wave function $\widetilde{\psi}_d(\vec{P})$ is very narrow so that $\left|\widetilde{\psi}_d(\vec{P})\right|^2$ is approximately a delta function. For simplicity we also examine the case that the initial momentum of decaying state is approximately zero. This allows us to write:

$$\left|\widetilde{\psi}_d(\vec{P})\right|^2 = \left|\widetilde{\psi}_d(\vec{p}_1+\vec{p}_2)\right|^2 \approx \delta^3(\vec{p}_1+\vec{p}_2). \tag{26}$$

Subsituting Eqs. (25) and (26) into eq. eqrefeq:prob-dist2 then gives this second version of Fermi's golden rule:

$$\gamma \;=\; \frac{2\pi}{\hbar}\int d^3p \, |V_{\vec{p}_1,\vec{p}_2,d}|^2 \, \delta(E_d-E_p)\delta^3(\vec{p}_1+\vec{p}^2). \tag{27}$$

It is reassuring that the addition of momentum conservation has little effect on Fermi's golden rule. In addition to requiring the replacement of the initial state with an infinite-volume state normalized to a delta function in momentum in the definition of $V_{\vec{p}_1,\vec{p}_2,d}$ and the extraction of a momentum-conserving delta function in Eq. (26), we need only add a momentum-conserving delta function to the standard formula for the decay rate.

Note, it would not be difficult to generalize this further to the case where the initial state had an non-zero average momentum $\langle\vec{P}\rangle$.