Physics GR6038 | Professor Christ
due 2/01/2026

Problem Set #12

Problem 46 has been divided into two problems to properly weight its grade.
Don't miss the third and fourth problems on the next page.
Extraneous 3/2 removed from problem 47.

44. Three spin-1 particles with angular momentum operators $\vec{S}_1$, $\vec{S}_2$ and $\vec{S}_3$ are located at fixed positions . These three spins interact through the Hamiltonian

$$H = \lambda \left(\vec{S}_1 \cdot \vec{S}_2 + \vec{S}_1 \cdot \vec{S}_3 + \vec{S}_2 \cdot \vec{S}_3 \right)$$

(a) Find the allowed energies of this systems and their multiplicities. Make a simple table showing each energy and its multiplicity.

(b) Assume that each of these three spins has a magnetic dipole moment $\vec{\mu}^{(i)} = \gamma \vec{S}^{(i)}$. Apply a magnetic field so that the magnetic energy:

$$H_{\mathrm{B}} = -\gamma \vec{B} \cdot \left(\vec{S}_1 + \vec{S}_2 + \vec{S}_3 \right),$$

is added to the Hamiltonian. What are the allowed energies and their multiplicities?

(c) Finally consider the case that the gyromanetic ratio γ_1 for particle 1 is different from that of particles 2 and 3 so that

$$H_{\mathrm{B}} = -\vec{B} \cdot \left(\gamma_1 \vec{S}_1 + \gamma_{23} (\vec{S}_2 + \vec{S}_3) \right),$$

Find the resulting spectrum of $H + H_{\mathrm{B}}$ in the limits of weak and strong $\vec{B}$ and describe the corresponding eigenstates. For the weak field case work to first order in $\vec{B}$ and for the strong field case work to first order in λ.

45. Consider a quantum mechanical system with a Hamiltonian H at a temperature T described by a density matrix ρ. The eigenstates and eigenvalues of H are $|\phi_n\rangle$ and E_n. (Since n labels the states, there may be many values of n whose values for E_n are the same.)

(a) Write down an explicit expression for ρ in terms of $|\phi_n\rangle$, E_n.

(b) Write down an explicit expression for the von Neumann entropy σ in terms E_n.

(c) If the Helmholz free energy F is defined as $F = E - kT\sigma$ where E is the total energy of the system, write down an explicit expression for F in terms of E_n.

46. (From Gottfried and Yan, Ch 2, problem 4) Consider a system S composed of two subsystems A and B and let $\rho^{A \otimes B}$ be the density matrix for a configuration of the combined system S. Show that for the reduced density matrix ρ^A for subsytem A to describe a pure state then the density matrix $\rho^{A \otimes B}$ must have the form:

$$\rho^{A \otimes B} = |\phi\rangle_A \, {}_A\langle\phi| \otimes \rho^B$$

where $|\phi\rangle_A$ is a normalized state in the Hilbert space for the system A and ρ^B is the reduced density matrix for the subsystem B.

47. A quantum system is made up of the product of two spin-$\frac{1}{2}$ states: $|\pm\rangle_A \otimes |\pm\rangle_B$, where the factor $|\pm\rangle_{A/B}$ describes a state with $s_z = \pm\frac{\hbar}{2}$. Let $\vec{s}^{\,A}$ and $\vec{s}^{\,B}$ be the corresponding spin operators and assume a Hamiltonian of the form:

$$H = \frac{\alpha kT}{\hbar} S_z + \frac{\beta kT}{2} \frac{\vec{S}^{\,2}}{\hbar^2}$$

where $\vec{S} = \vec{s}^{\,A} + \vec{s}^{\,B}$ and the factors of $\frac{1}{2}$ and kT have been introduced to give simple expressions for the resulting Boltzmann factors.

(a) If this system is in a mixed state at the temperature T write down the 4×4 density matrix $\rho^{A\otimes B}$ describing that system. (This can be expressed as a sum over four projection operators with explicit coefficients.)

(b) What is the average result if the total spin $\vec{S}$ is measured many times on such a system?

(c) Find the reduced 2×2 density matrix ρ^A that can be used to calculate the results from measuring operators O^A which act on only the states $|\pm\rangle^A$ that appear in the basis states $|\pm\rangle_A \otimes |\pm\rangle_B$.

(d) If only the spin $\vec{s}^A$ is measured multiple times, use your result from part (c) to determine the average result? Does $\langle \vec{S}^{A\otimes B} \rangle = 2\langle \vec{s}^A \rangle$?

48. Use the results of problem 47 to answer the following quesitons.

(a) If subsystem associated with the states $|\pm\rangle$ is interpreted as a system with the Hamiltonian $H = \frac{\alpha}{\hbar} s_z^A$ what temperature would you assign to the system?

(b) What is the von Neumann entropy $\sigma^{A\otimes B}$ of the system described by the density matrix $\rho^{A\otimes B}$ obtained in part (a)?

(c) What is the von Neumann entropy σ^A of the system described by the reduced density matrix ρ^A obtained in part (c)? What is the relation between the magnitudes of the von Neumann entropies $\sigma^{A\otimes B}$ and sum of the entropies of the subsystems A and B which because of symmetry equals $2\sigma^A$?