Physics GR6038 — Professor Christ

due 02/8/2026

Problem Set #13

49. Return to problem 29 in problem set #8 from last semester and express the time-dependent Schrödinger equation is terms of the applied frequency, here called $\omega$, and the Larmor frequencies $\omega_0 = \gamma B_0$ and $\omega_1 = \gamma B_1$ at which the electron would precess if only the original field $B_0\hat{z}$, or a time-independent version of the rotating field $B_1$, were present:

$$i\frac{d}{dt}|\psi(t)\rangle = -\Big\{\frac{\omega_0}{2}\sigma_3 + \frac{\omega_1}{2}\Big(\sigma_1\cos(\omega t) - \sigma_2\sin(\omega t)\Big)\Big\}|\psi(t)\rangle.$$

(a) Use Python to evaluate the analytic solution you obtained when working problem 29. You should choose time units so that $\omega_0 = 2\pi$, making the period of Larmor precession equal to 1 in those units. Use the initial state $(|+\frac{1}{2}\rangle + |-\frac{1}{2}\rangle)/\sqrt{2}$, $\omega_1 = \omega_0/5$ and plot the expectation value for the three components of the electron's spin as a function of time $t$ for $0 \le t \le 5$. Plot the graph using 5,000 values for the time $t$ separated by a time interval $\Delta t = 0.001$ for the value $\omega = 2$. (Only this final plot and the second one requested below need to be included when you submit your solution.)

(b) Show that the time-dependent Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle.$$

is solved by the $\Delta t \to 0$ limit of the product

$$|\psi(t)\rangle = \prod_{n=0}^{N-1} e^{-iH(t_n)\Delta t/\hbar}|\psi(0)\rangle.$$

where $N = t/\Delta t$ and $t_n = n\Delta t$ and the factors in the product are ordered so that those at later time appear farther to the left. With what power of $\Delta t$ do the errors in $|\psi(t)\rangle$ in the above product vanish as $\Delta t \to 0$?

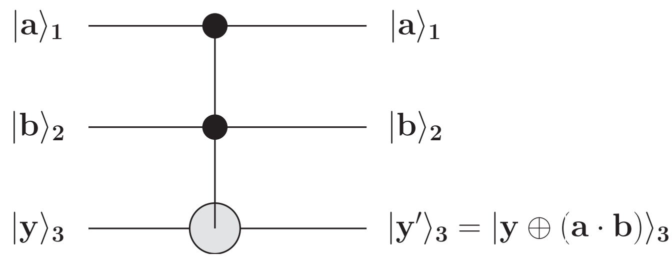
(c) Can you find a variant of this method with smaller errors that can be evaluated with the same or less computational cost? (You might take inspiration from the leap-frog integration method introduced in the Schrodinger.ipynb example posted on the web.)

(d) Use the method in part b) to obtain an alternative numerical solution to the specific problem posed in part a). Include with your problem solutions a graph of this numerical solution using the same time step $\Delta t = 0.001$ and time interval $0 \le t \le 5$ as used in part a).

[It may be helpful to begin with a working sample Jupyter notebook larmor.ipynb, posted on the course web site, which addresses simple Larmor precession and then to modify it to solve this homework problem.]

50. Show that a two-input and two-output quantum gate for which one of the outputs is the AND of its two inputs cannot be realized.

51. Find the $8 \times 8$ unitary matrix that describes the action of the Toffoli gate shown in the figure. Assign the number $n = a + 2b + 4y$, $0 \le n < 8$ to the basis state $|a\rangle_1 \otimes |b\rangle_2 \otimes |y\rangle_3$ and label the columns and rows of the $8 \times 8$ matrix to the basis vector associated with the number $n$, where $n$ is assigned in ascending order moving from left to right and top to bottom respectively. Here $a$, $b$ and $y$ take the values 0 and 1. For the labeling the rows, the state $|y\rangle_3$ should be replaced by the state $|y'\rangle_3$. (Here the logical product $a \cdot b$ indicates the AND of $a$ and $b$.)




52. Design a quantum circuit that will teleport a two-Qbit quantum state (i.e. a general state in the four-dimensional complex vector space of sums of products of pairs of Qbits) between two distant observers: a "two-Qbit teleporter". Prove that your design works.

    How would you generalize your two-Qbit teleporter into an $N$-Qbit teleporter? How many classical binary bits must be exchanged in your $N$-Qbit teleporter between the remote systems?