Physics GR6038 Professor Christ

due 03/01/2026

Problem Set #16

Don't miss problem 63 on the second page

61. A negatively charged muon with mass $m_\mu = 106$ MeV$/c^2$ and an electron with mass $m_e = 0.51$ MeV$/c^2$ orbit a helium nucleus. Initially the muon is in the $2p$, $m_l = 0$, state while the electron is in the $1s$ state. Compute the rate for the decay process in which the muon falls to the $1s$ state while the electron is ejected from the atom. Since $m_\mu \gg m_e$, the Helium nucleus and the muon occupy a very small volume and appear to the electron as an accurate approximation to a hydrogen nucleus. Call this the central charge approximation.

   (a) Treat the deviation between the charge distribution of the muon and this central charge approximation as creating a perturbing electrostatic potential $V(\vec{r})$ which acts on the electron. Write down an expression for the decay rate in terms of a matrix element of $V(\vec{r})$ between the initial and final states where the final state is the product of the 1s muon and a plane wave for the ejected electron.

   (b) Determine this matrix element by first evaluating the factor in the matrix element which involves the muon wave function.

   (c) Second evaluate the portion of matrix element involving the electron's coordinates.

   (d) Finally evaluate the decay rate by integrating over the direction of the emitted electron.

62. Consider two degenerate states $|d_1\rangle$ and $|d_2\rangle$ which are coupled to a continuum of states $|\phi_{\vec{p}}\rangle$ by a potential V:

$$\langle \phi_{\vec{p}} | V | d_i \rangle = V_{\vec{p},i}$$

where $H = H_0 + V$ with

$$\begin{aligned} H_0 |d_i\rangle &= E_d |d_i\rangle \quad i = 1, 2 \\ H_0 |\phi_{\vec{p}}\rangle &= E_{\vec{p}} |\phi_{\vec{p}}\rangle \end{aligned}$$

and we assume that $\langle d_i | V | d_j \rangle = \langle \phi_{\vec{p}} | V | \phi_{\vec{p}'} \rangle = 0$. Show using the Wigner-Weisskopf method that the state $\beta_1 |d_1\rangle + \beta_2 |d_2\rangle$ decays with the exponent $-\gamma t/2$ provided the quantities β_1, β_2 and γ satisfy the following two conditions:

   (a) The column vector $\begin{pmatrix} \beta_1 \\ \beta 2 \end{pmatrix}$ is an eigenvector of the 2×2 matrix

$$\begin{aligned} \Gamma_{i,j} &= \frac{2\pi}{\hbar} \int d^3p \, \delta(E_{\vec{p}} - E_d) \langle d_i | V | \phi_{\vec{p}} \rangle \langle \phi_{\vec{p}} | V | d_j \rangle \\ &\quad + \frac{2i}{\hbar} \mathcal{P} \left\{ \int d^3p \, \frac{\langle d_i | V | \phi_{\vec{p}} \rangle \langle \phi_{\vec{p}} | V | d_j \rangle}{E_d - E_{\vec{p}}} \right\} \end{aligned}$$

   (b) The quantity γ is the corresponding complex eigenvalue.

63. Consider a nucleus which can be in two states: the ground state $\big|B(\vec{P}_B)\big\rangle$ and an excited state $\big|A(\vec{P}_A)\big\rangle$ with momenta and energies $\vec{P}_B$, $\frac{\vec{P}_B^2}{2M_B}$ and $\vec{P}_A$, $\frac{\vec{P}_A^2}{2M_A} + \Delta E$ respectively. The excited state A can decay into the ground state B with the emission of a photon with momentum $\vec{k}$ and energy $|\vec{k}|c$ represented by the state $\big|\gamma(\vec{k})\big\rangle$. For simplicity we will assume the photon is a spin-0 particle with no polarization. All momenta are continuous, infinite-volume variables and the states obey:

$$\begin{aligned} \big\langle B(\vec{P}\,')\big|B(\vec{P})\big\rangle &= \big\langle A(\vec{P}\,')\big|A(\vec{P})\big\rangle = \delta^3(\vec{P}\,' - \vec{P}) \qquad (1) \\ \big\langle\gamma(\vec{k}\,')\big|\gamma(\vec{k}\,)\big\rangle &= \delta^3(\vec{k}\,' - \vec{k}). \qquad (2) \end{aligned}$$

The decay $A \to B + \gamma$ is caused by the non-zero matrix element of the potential V_{op} connecting the unstable excited state $\big|A(\vec{P}_A)\big\rangle$ with the final product state $|\gamma(\vec{k})\rangle \otimes |B(\vec{P}_B)\rangle$. If the three states $|A\rangle$, $B\rangle$ and $|\gamma\rangle$ are described the wave functions $\psi_A(\vec{r})$, $\psi_B(\vec{r})$ and $\psi_\gamma(\vec{r})$ then the matrix element of V_{op} between these states is given by:

$$[\langle\gamma| \otimes \langle B|]\, V_{\text{op}}|A\rangle = V \int d^3r\, \psi_\gamma(\vec{r})^* \psi_B(\vec{r})^* \psi_A(\vec{r}) \qquad (3)$$

(a) Show that the equation

$$\Big[\big\langle\gamma(\vec{k})\big| \otimes \big\langle B(\vec{P}_B)\big|\Big] V_{\text{op}}\big|A(\vec{P}_A)\big\rangle = V_{\gamma BA}\delta^3(\vec{k} + \vec{P}_B - \vec{P}_A). \qquad (4)$$

is obeyed and find $V_{\gamma BA}$.

(b) Determine the rate γ for the decay of the unstable state $\big|A(\vec{P}_A = \vec{0})\big\rangle$ at rest to lowest order in V_{AB}. Assuming that $\Delta E \ll M_B c^2$, find the approximate reduction in the final photon energy that results when the energy of the recoiling final-state nucleus B is taken into account.