Physics GR6038 Professor Christ

due 03/08/2026

Problem Set #17

Problem 64 reworded and the order of 66 and 67 reversed

64. This is a continuation of problem 63 in problem set #16 in which the initial and final nuclear states are changed from plane wave states to states confined to move in a deep, three-dimensional square-well potential so that $0 \le x_i \le L$ where the x_i for $1 \le i \le 3$ are the three coordinates determining the position of either the initial or the final nucleus. As in problem 63, the emitted photon is still described by a delta-function-normalized plane wave state:

$$\langle \vec{r} | \psi_\gamma(\vec{k}\,) \rangle = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} e^{i\vec{k}\cdot\vec{r}/\hbar}$$

and the potential $V_{\rm op}$ causing the decay has the matrix element:

$$[\langle \gamma | \otimes \langle B |]\, V_{\rm op} | A \rangle = V \int d^3 r\, \psi_\gamma(\vec{r})^* \psi_B(\vec{r})^* \psi_A(\vec{r}) \tag{1}$$

where V is a constant.

(a) Determine the possible energy eigenstates for the initial and final nuclei constrained to move in this potential well? Identify these states with three integers $n_i > 0$, $1 \le i \le 3$

(b) If the initial nucleus is in the state of lowest kinetic energy, determine the possible final-state photon energies that might be observed as a function of the energy difference ΔE that is defined in problem 63. To answer this question you may ignore the size of the corresponding decay amplitude.

(c) Assuming that the photon momentum $\vec{k}$ obeys $|\vec{k}\,| \ll \hbar/L$, what photon energies will be seen in the final state if the matrix element for the $A \to B + \gamma$ decay is evaluated to zeroth order in $\vec{k}$ for the case that the initial nucleus is in the state of lowest kinetic energy? (Now we are asking for which momenta is the decay amplitude non-zero.)

(d) Find the differential decay rate $d^3\gamma/(d\Omega_{\hat{k}} dk)$ to second order in V for this decay when computed also to second order in the components of $\vec{k}$ for the case that the final nucleus is in the spatial state determined by the three integers (n_1, n_2, n_3) where the initial nucleus is in its ground state but the final nucleus is not.

65. N spinless bosons of mass m move in the three-dimensional harmonic oscillator potential

$$V(r) = \frac{1}{2} m\omega^2 \vec{r}^{\,2}.$$

Find the energies and degeneracies of the three lowest energy levels.

66. Consider a system of identical bosons confined to a volume V. Let $\{\phi_k(\vec{r}\,)\}$ be a complete, orthonormal set of single-particle wave functions. If $a_k^\dagger$ is the creation operator for the state ϕ_k in our second-quantized formalism, define the operator

$$\Psi(\vec{r}\,) = \sum_k a_k \phi_k(\vec{r}\,).$$

(a) Show that when applied to the vacuum, the operator

$$\int_V f(\vec{r})\Psi^\dagger(\vec{r})d^3r$$

creates a single particle state with wave function $f(\vec{r})$.

(b) Consider an operator $\mathcal{V}$ which when acting on N-particle states has the form

$$V_N = \frac{1}{2}\sum_{j=1}^{N}\sum_{\substack{i=1\\ i\neq j}}^{N} V(\vec{r}_i, \vec{r}_j)$$

where $\vec{r}_1 \ldots \vec{r}_N$ are the coordinates operators for the N particles and $V(\vec{r}, \vec{r}\,')$ is a real function of the variables $\vec{r}$ and $\vec{r}\,'$. Using the second quantized formalism, $\mathcal{V}$ can be written:

$$\mathcal{V} = \frac{1}{2}\int d^3r \int d^3r' \Psi^\dagger(\vec{r}\,')\Psi^\dagger(\vec{r})V(\vec{r}, \vec{r}\,')\Psi(\vec{r}\,')\Psi(\vec{r}).$$

Show that this formula is correct for matrix elements of the form:

$$\langle n_1', n_2', \ldots n_k' \ldots |\mathcal{V}| n_1, n_2, \ldots n_k \ldots\rangle$$

for the following two cases:

i. When $n_k' = n_k$ for all states k.

ii. When $n_k' = n_k$ for all states k except for the four distinct states: $k = a, b, c, d$ and for these cases:

$$n_a' = n_a + 1, \quad n_b' = n_b + 1, \quad n_c' = n_c - 1, \quad n_d' = n_d - 1$$

(c) If the term $i = j$ is included in the definition of V_N what is the corresponding formula for $\mathcal{V}$?

67. A system of N spinless bosons in three dimensions is confined inside a cube of side L by the potential

$$V(x, y, z) = \begin{cases} 0 & \text{if } \; 0 \le x \le L, \; 0 \le y \le L, \; 0 \le z \le L \\ \infty & \text{otherwise.} \end{cases}$$

(a) If each pair of bosons interacts through the potential

$$V(\vec{r}, \vec{r}\,') = \lambda\delta^3(\vec{r} - \vec{r}\,'),$$

use the expression given in Problem 66. b) to write down the Hamiltonian for the system using the “second quantized” field operator $\Psi(\vec{r})$ and its hermitian conjugate $\Psi(\vec{r})^\dagger$ which destroy and create a particle at the position $\vec{r}$.

(b) What is the ground state and ground state energy to 0^{th} order in λ for the Hamiltonian of (a)?

(c) What is the shift in the ground state energy to first order in λ?