

Problem Set #18

68. Consider a system of N , non-interacting, spin-1/2 fermions. If $\{\phi_k(\vec{r}, m_s)\}$ is a complete set of single particle energy eigenstates with energies $\{E_k\}$ and a_k is the usual destruction operator for the state ϕ_k , show that the “second quantized” operator

$$\Psi(\vec{r}, m_s, t) = \sum_k a_k \phi_k(\vec{r}, m_s) e^{-iE_k t/\hbar}$$

obeys

$$i\hbar \frac{\partial}{\partial t} \Psi(\vec{r}, m_s, t) = -[H, \Psi(\vec{r}, m_s, t)]$$

where H is the complete Hamiltonian for the N -particle system. (Note, that this equation involves a commutator even though the operators a_k and $a_k^\dagger$ obey anti-commutation relations.) If the single particle Hamiltonian is $\vec{p}^2/2m + V(\vec{r})$, show that Ψ obeys

$$i\hbar \frac{\partial}{\partial t} \Psi(\vec{r}, m_s, t) = \left[-\frac{\hbar^2}{2m} \vec{\nabla}^2 + V(\vec{r}) \right] \Psi(\vec{r}, m_s, t).$$

69. (Baym, Chapter 19, Problem 2)
- Find a general formula, accurate to first order in the inter-particle interaction $V(\vec{r} - \vec{r}')$ for the energy of an $N + 1$ particle system of spin-1/2 fermions with one particle of momentum $\vec{p}$ outside of an N -particle Fermi sea. Your answer should be expressed as the difference of the energy of the specified state and the energy of the N -particle ground state. (This might be most easily solved by using creation and annihilation operators to express the difference as the matrix element of a difference of operators—a difference which can be simplified using anti-commutation relations.)
 - Evaluate your result explicitly for the case of a Coulomb interaction (include the effects of a uniform positive charge density making the whole system neutral).

70. Consider a system of N identical spin-1/2 particles moving independently in a three-dimensional harmonic potential. The total Hamiltonian is:

$$H = \sum_{i=1}^N \left(\frac{\vec{p}_i^2}{2m} + \frac{1}{2} \omega^2 \vec{r}_i^2 \right).$$

- (a) Find (but do not solve) an equation relating N and the energy of the ground state and determine the degeneracy of the ground state.
- (b) If the particles have magnetic moment $\vec{\mu} = \gamma \vec{s}$ and the system is placed in a uniform magnetic field $\vec{B}$, find the energy of the ground state, E_{gnd} as a function of B . Draw rough graphs of your result for E_{gnd} for fixed, weak B as a function of N and for fixed N as a function of B . Be careful that as B increases you are always dealing with the lowest energy state. (You need only give a specific answer when $|\vec{B}| < 2\omega/\gamma$.)