

Problem Set #24

85. Generalize the analysis given in class of the resonance-decay problem to the case where the decaying state has a general integer angular momentum l :

- Express $H = H_0 + V$.
- Assume H_0 has two types of eigenstates: $\{|m\rangle\}_{-l \leq m \leq l}$ and $|\phi_{\vec{p}}\rangle$ with:

$$H_0|m\rangle = E_d|m\rangle \quad H_0|\phi_{\vec{p}}\rangle = \frac{p^2}{2M}|\phi_{\vec{p}}\rangle,$$

where $|m\rangle$ and $|\phi_{\vec{p}}\rangle$ obey the normalization conditions:

$$\langle m'|m\rangle = \delta_{m',m} \quad \langle \phi_{\vec{p}'}|\phi_{\vec{p}}\rangle = (2\pi\hbar)^3 \delta(\vec{p}' - \vec{p}).$$

- Assume that the only non-zero matrix elements V are

$$V_{m,\vec{p}}^* = V_{\vec{p},m} = \langle \phi_{\vec{p}}|V|m\rangle \equiv F(p)Y_{l,m}(\hat{p}).$$

- Find the incoming states $|\psi_{\vec{p}}^{in}\rangle$ in terms of $|m\rangle$ and $|\phi_{\vec{p}}\rangle$.
 - Calculate the scattering amplitude $f(\theta, \phi)$.
 - Find the scattering phase shift $\delta_l(p)$.
86. Using stationary state scattering theory, consider scattering from a hard sphere of radius a . In contrast with the low energy limit discussed in class to which only the s -wave phase shift contributes, explore the high-energy limit in which a sum over the contributions of many partial waves must be evaluated. You might do this using a simple Python program. Examine the particular case where the input momentum is $400\hbar/a$. What is your result for the total cross section at this incident momentum and determine how many terms are needed in the partial wave sum to obtain a result with a accuracy less than 1%?
87. Find the average electric field at the point $\vec{r}$ and time t for each of the following initial states described at $t = 0$:

(a) $a_i(\vec{k}_{\vec{j}})^\dagger|0\rangle$

(b) $\frac{1}{\sqrt{1+|b|^2}}\{|0\rangle + b a_i(\vec{k}_{\vec{j}})^\dagger|0\rangle\}$

(c) $\sum_{m=0}^{\infty} \frac{b^m}{m!} \{a_i(\vec{k}_{\vec{j}})^\dagger\}^m|0\rangle e^{-|b|^2/2}.$

88. Consider an electromagnetic field obeying periodic boundary conditions on the walls of a cube of volume V , described by the vector potential $\vec{A}(\vec{r}, t)$ satisfying the Coulomb gauge condition $\vec{\nabla} \cdot \vec{A} = 0$. Given:

$$\begin{aligned}\vec{A}(\vec{r}) &= \sum_{j,n} \mathcal{E}_j(\vec{k}_n) \left(q_j^{re}(\vec{k}_n) + i q_j^{im}(\vec{k}_n) \right) \frac{e^{i\vec{k} \cdot \vec{r}}}{\sqrt{V}} \\ p_j^{re}(\vec{k}_n) &= \frac{1}{2\pi c^2} \frac{d}{dt} q_j^{re}(\vec{k}_n), \quad p_j^{im}(\vec{k}_n) = \frac{1}{2\pi c^2} \frac{d}{dt} q_j^{im}(\vec{k}_n) \\ a_j(\vec{k}_n) &= \frac{1}{2} \left\{ \sqrt{\frac{k_n}{2\pi\hbar c}} \left(q_j^{re}(\vec{k}_n) + i q_j^{im}(\vec{k}_n) \right) + i \sqrt{\frac{2\pi c}{\hbar k_n}} \left(p_j^{re}(\vec{k}_n) + i p_j^{im}(\vec{k}_n) \right) \right\},\end{aligned}\tag{1}$$

derive each of the formulae:

$$\begin{aligned}\vec{B}(\vec{r}) &= i \sum_{j,n} \sqrt{\frac{2\pi\hbar c}{V k_n}} \vec{k}_n \times \hat{\mathcal{E}}_j(\vec{k}_n) \left\{ a_j(\vec{k}_n) e^{i\vec{k}_n \cdot \vec{r}} - a_j(\vec{k}_n)^\dagger e^{-i\vec{k}_n \cdot \vec{r}} \right\} \\ \vec{E}(\vec{r}) &= i \sum_{j,n} \sqrt{\frac{2\pi\hbar c}{V k_n}} |\vec{k}_n| \hat{\mathcal{E}}_j(\vec{k}_n) \left\{ a_j(\vec{k}_n) e^{i\vec{k}_n \cdot \vec{r}} - a_j(\vec{k}_n)^\dagger e^{-i\vec{k}_n \cdot \vec{r}} \right\} \\ H &= \frac{1}{8\pi} \int d^3r \left\{ \vec{E}^2(\vec{r}) + \vec{B}^2(\vec{r}) \right\} = \sum_{j,n} \hbar \omega \left\{ a_j(\vec{k}_n)^\dagger a_j(\vec{k}_n) + \frac{1}{2} \right\} \\ \vec{P} &= \frac{1}{4\pi c} \int d^3r \vec{E}(\vec{r}) \times \vec{B}(\vec{r}) = \sum_{j,n} \hbar \vec{k}_n a_j(\vec{k}_n)^\dagger a_j(\vec{k}_n).\end{aligned}$$

All of the equations above can be considered as time-independent operator equations with the exception of Eqs. (1) which should be viewed as classical equations showing how the $\dot{q}_i^{re}(\vec{k}_n)$ and $\dot{q}_i^{im}(\vec{k}_n)$ appearing in $\vec{E}$ can be replaced by the canonical momenta variables $p_i^{re}(\vec{k}_n)$ and $p_i^{im}(\vec{k}_n)$.

89. **(This is an extra problem which will not be included in your homework grade.)** Two classical magnetic moments $\vec{\mu}_1$ located at $\vec{r}_1$ and $\vec{\mu}_2$ located at $\vec{r}_2$ interact with the quantum electromagnetic field through the Hamiltonian:

$$H = -\vec{\mu}_1 \cdot \vec{B}(\vec{r}_1) - \vec{\mu}_2 \cdot \vec{B}(\vec{r}_2)$$

- Express the magnetic field $\vec{B}(\vec{r})$ above in terms of the creation and annihilation operators $a_i(\vec{k}_l)$ and $a_i^\dagger(\vec{k}_l)$ for photons with wave number $\vec{k}_l$ and polarization $\hat{\mathcal{E}}_i(\vec{k}_l)$ in a finite volume of side L .
- Using your expression from part (a), write down the complete Hamiltonian for the electromagnetic degrees of freedom in terms of the creation and annihilation operators.
- Find the energy eigenvalues and eigenstates of the resulting Hamiltonian. In particular, find the dependence of the resulting ground state energy on $\vec{\mu}_1$, $\vec{\mu}_2$ and $\vec{r}_2 - \vec{r}_1$ in the limit $L \rightarrow \infty$. [Hint: This can be done by using new annihilation and creation operators obtained by shifting $a_i(\vec{k}_l)$ and $a_i^\dagger(\vec{k}_l)$ by complex constants].

[In Coulomb gauge all magnetic forces must come from the exchange of quantum photons and this problem shows how this can work.]