Physics GR6038 | Professor Christ
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Relation between the cross section $d\sigma/d\Omega$ and $f(\theta,\phi)$ in stationary state scattering theory

Here we provide a wave-packet derivation of the standard formula:

$$\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2. \tag{1}$$

The scattering amplitude $f(\theta,\phi)$ appears in the asymptotic form of the in-state Schrodinger equation solution $\psi^{\rm in}_{\vec p}(\vec r)$ when $r = |\vec r|$ becomes large:

$$\psi^{\rm in}_{\vec p}(\vec r) \sim e^{i\vec p\cdot\vec r/\hbar} + f(\theta,\phi)\frac{e^{ipr/\hbar}}{r}. \tag{2}$$

Begin by creating a familiar Gaussian (or similar) wave packet from free plane wave states:

$$F(\vec r) = \int d^3p \frac{1}{(2\pi\hbar)^{3/2}} e^{i\vec p\cdot\vec r/\hbar}\widetilde{F}(\vec p) \tag{3}$$

where, for the Gaussian case we use:

$$\widetilde{F}(\vec p) = \left(\frac{1}{\Delta p\sqrt{\pi}}\right)^{3/2} \exp\{-(\vec p - \vec p_0)^2/2\Delta p^2\}. \tag{4}$$

Thus, we assume that $\widetilde{F}(\vec p)$ describes a free wave packet moving with a central momentum $\vec p_0$ and a width at $t=0$ of Δp. We next combine the amplitude $\widetilde{F}(\vec p)$ with our eigenstates of the complete scattering Hamiltonian to form a wave packet with the correct time dependence:

$$\psi(\vec r, t) = \frac{1}{(2\pi\hbar)^{3/2}}\int d^3p\; \widetilde{F}(\vec p)\; \psi^{\rm in}_{\vec p}(\vec r) e^{-i\frac{p^2}{2m}t/\hbar} \tag{5}$$

We must now try to understand the behavior of this exact solution to the complete time-dependent Schrödinger equation and deduce from its behavior at large times an expression for the scattering cross section.

For the time t large and negative, the $\exp(-ip^2t/2m\hbar)$ will cause the integral over $\vec p$ in Eq. 5 to oscillate rapidly and lead to a vanishing small value of $\psi(\vec r, t)$ unless the other parts of the integral contain a compensating rapid variation of phase. This cannot come from the factor $\widetilde{F}(\vec p)$ since this is real. However, the in-state wave function $\psi^{\rm in}_{\vec p}(\vec r)$ can show such a variation which can be made equally rapid for appropriately large values of r. For such values of r, we can use the in-state's asymptotic form given in Eq. 2. For large negative t, the only compensating phase can come from the plane wave term, $\exp(+i\vec{p}\cdot\vec{r}/\hbar)$. As usual, we can impose the requirement of stationary phase for the central value of $\vec{p}=\vec{p}_0$:

$$\nabla_p\left(+\vec{p}\cdot\vec{r}-t\frac{p^2}{2m}\right)_{\vec{p}=\vec{p}_0}=0 \tag{6}$$

or

$$\vec{r}=t\frac{\vec{p}_0}{m}, \tag{7}$$

the usual classical motion expected of our wave packet. Thus, for large negative time our solution $\psi(\vec{r},t)$ behaves exactly like an incoming wave packet constructed from the Fourier amplitude $\widetilde{F}(\vec{p})$ and the plane wave piece $\exp(+i\vec{p}\cdot\vec{r}/\hbar)$ — the other, outgoing piece has a phase which oscillates with the wrong sign and does not contribute.

For large positive t the above description of the plane wave continues to be true with Eq. 7 obeyed. With large positive t the solution position $\vec{r}$ will now lie on the other side of the scattering center and represents the unscattered wave packet moving on. However, with t positive, cancelation is now possible between the positive phase $+ipr/\hbar$ of the outgoing spherical wave $\exp(ipr/\hbar)/r$ and the negative phase $-ip^2t/2m\hbar$. Thus, we expect an outgoing wave packet which looks like a spherical shell with approximate radial location given by the stationary phase condition:

$$\nabla_p\left(+pr-t\frac{p^2}{2m}\right)_{\vec{p}=\vec{p}_0}=0 \tag{8}$$

or

$$r=t\frac{p_0}{m}. \tag{9}$$

Next we will compute the probability, dP of finding the scattered particle coming outward from the scattering center within a solid angle $d\Omega$:

$$dP=\int_{r_{min}}^{r_{max}}r^2drd\Omega\left|\frac{1}{(2\pi\hbar)^{3/2}}\int d^3p\widetilde{F}(\vec{p})f(\theta,\phi)\frac{e^{ipr/\hbar}}{r}e^{-i\frac{p^2}{2m}t/\hbar}\right|^2 \tag{10}$$

where the limits of radial integration, r_{max} and r_{min} are chosen to encompass the radial location of the outgoing shell of probability at the time t. Here the element of solid angle $d\Omega$ corresponds to the angles θ and ϕ. We can cancel the factors of r^2, extract the factor of $f(\theta,\phi)$ by assuming that it varies little as the momentum $\vec{p}$ is integrated over the width of the wave function $\widetilde{F}(\vec{p})$ so this equation becomes:

$$dP=d\Omega\,|f(\theta,\phi)|^2\int_{r_{min}}^{r_{max}}dr\left|\frac{1}{(2\pi\hbar)^{3/2}}\int d^3p\widetilde{F}(\vec{p})e^{ipr/\hbar}e^{-i\frac{p^2}{2m}t/\hbar}\right|^2. \tag{11}$$

The three dimensional momentum integral inside the squared modulus can be related to the Fourier transform of $\widetilde{F}(\vec{p})$:

$$F(\vec{r},t)=\frac{1}{(2\pi\hbar)^{3/2}}\int d^3pe^{i\vec{p}\cdot\vec{r}}\widetilde{F}(\vec{p})e^{-i\frac{p^2}{2m}t/\hbar}. \tag{12}$$

First we express the momentum $\vec{p}$ in terms of its components perpendicular and parallel to $\vec{p}_0$: $\vec{p} = \vec{p}_\perp + p_\parallel \hat{p}_0$. Next the variable $p = \sqrt{\vec{p}^2} = \sqrt{\vec{p}_\perp^2 + p_\parallel^2}$ can be expanded to zero order in the small perpendicular components $\vec{p}_\perp$:

$$p \approx p_\parallel . \tag{13}$$

Since in this approximation the perpendicular components $\vec{p}_\perp$ do not appear in the exponent, the integral over $\vec{p}_\perp$ appearing in Eq. 11 is simply the Fourier transform with respect to $\vec{p}_\perp$ evaluated at $\vec{r}_\perp = 0$. Finally we can evaluate the integral over $p_\parallel$ by simply recognizing it as transforming to the Fourier transform variable $r_\parallel = r$:

$$dP = d\Omega \, |f(\theta,\phi)|^2 \int_{r_{min}}^{r_{max}} dr \, |F(\vec{r}_\perp = 0, r, t)|^2 \, . \tag{14}$$

This expression can be easily recognized if we observe that the probability for the incident particle striking a cross sectional area $d\sigma$ located at the origin is simply the probability of finding the incident particle in a small cylinder of cross sectional area $d\sigma$ parallel to $\vec{p}_0$ extending throughout the wave function $F(\vec{r}, t)$:

$$dP = d\sigma \int_{r_{min}}^{r_{max}} dr \, |F(\vec{r}_\perp = 0, r, t)|^2 \, . \tag{15}$$

The volume of interest is the darkly shaded volume in the diagram below. Comparing Eqs. 14 and 15, we recognize that the cross section is given by

$$d\sigma = d\Omega \, |f(\theta,\phi)|^2 \tag{16}$$

which is the result we are trying to establish.