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Definition of dimension

Here we attempt to give a correct proof that if we have a vector space V with a maximal set of N_a linearly independent vectors $\mathcal{A} = \{a_1, a_2, \ldots a_{N_a}\}$ then a second maximal set of linearly independent vectors $\mathcal{B} = \{b_1, b_2, \ldots b_{N_b}\}$ must contain the same number of vectors, *i.e.* $N_b = N_a$. This implies that the maximal number of vectors in any independent set has a single value N_a which can then be consistently identified as the dimension of V. Since a maximal set of linearly independent vectors is called a basis, we are showing that all sets of basis vectors have the same number of elements.

To show that $N_a = N_b$ we assume that $N_a > N_b$ and show that this is impossible. If $N_a \not> N_b$, we must conclude that $N_a \le N_b$. We can then exchange the roles of the two sets $\mathcal{A}$ and $\mathcal{B}$ and conclude that also $N_b \le N_a$ which implies $N_a = N_b$ the result we were trying to prove.

To show the inconsistency of the assumption $N_a > N_b$, we add the first vector a_1 in the set $\mathcal{A}$ to the set $\mathcal{B}$ and form the set of $N_b + 1$ vectors $\{a_1, b_1, b_d, \ldots, b_{N_b}\}$. Because the set $\mathcal{B}$ is maximal this new set which adds the vector a_1 to $\mathcal{B}$ must be linearly dependent so we can find a set of $N_b + 1$ coefficients $\{\widetilde{c}_1, \{c_i\}_{1 \le i \le N_b}\}$ with some of these coefficients non-zero which obeys:

$$\widetilde{c}_1 a_1 + c_1 b_1 + c_2 b_2, \ldots, c_{N_b} b_{N_b} = 0. \tag{1}$$

Since the elements of $\mathcal{B}$ are linearly independent, $\widetilde{c}_1$ must be non-zero. This implies that $\widetilde{c}_1 a_1$ is also non-zero so one or more of the coefficients $\{c_i\}_{1 \le i \le N_b}$ must also be non-zero. Relabel the N_b vectors $\{b_i\}_{1 \le i \le N_b}$ so that all of the non-zero coefficients $c_i \ne 0$ appear for the largest values of i. For later reference, we will call this step in which we show that one of the coefficients of the vectors b_k in Eq. (1) must be non-zero step #1.

Referring to these new labels, we then remove the vector b_{N_b} from the set $\{a_1, b_1, b_d, \ldots, b_{N_b}\}$. We label the new set:

$$\mathcal{B}_1 = \{a_1, b_1, b_d, \ldots, b_{N_b - 1}\}. \tag{2}$$

We can show that the set of vectors $\mathcal{B}_1$ is also a maximal, linearly independent set.

If $\mathcal{B}_1$ were linearly dependent, it is easy to see that we could express:

$$a_1 = d_1 b_1 + d_2 b_2 + \ldots d_{N_b - 1} b_{N_b - 1}. \tag{3}$$

We can then substitute this equation into Eq. (1) to eliminate the vector a_1. Then using a more compact notation, Eq. (1) becomes:

$$\widetilde{c}_1 \left\{ \sum_{i=1}^{N_b - 1} d_i b_i \right\} + \sum_{j=1}^{N_b} c_j b_j = 0. \tag{4}$$

This equation contradicts the assumption that the set $\mathcal{B}$ is linearly independent since we know that the coefficient c_{N_b} of the vector b_{N_b}, is non-zero by assumption. This step showing that the vectors in the new set $\mathcal{B}_1$ are linearly independent will be labeled step #2.

We can also show that the set $\mathcal{B}_1$ is a maximal set because for any vector $w \in V$ we can use the fact that $\mathcal{B}$ is maximal to write:

$$w = \sum_{i=1}^{N_b} f_i b_i. \tag{5}$$

Next we use Eq. (1) to express the vector b_{N_b} appearing in Eq. (5) in terms of the elements of $\mathcal{B}_1$

$$b_{N_b} = -\frac{1}{c_{N_b}} \{\widetilde{c}_1 a_1 + c_1 b_1 + c_2 b_2, \ldots, c_{N_b-1} b_{N_b-1}\} \tag{6}$$

We can then substitute this equation for b_{N_b} into Eq. (5). With the vector b_{N_b} replaced, Eq. (5) will then express an arbitrary vector w in terms of the vectors in the set $\mathcal{B}_1$. We will label this final step showing that $\mathcal{B}_1$ is maximal step #3.

We have thus removed one vector from $\mathcal{A}$ and created a new set $\mathcal{B}_1$ of N_b elements composed of one vector from $\mathcal{A}$ and $N_b - 1$ vectors from $\mathcal{B}$. Just as was the case for the set $\mathcal{B}$ the set $\mathcal{B}_1$ is a maximal independent set. If we remove a_1 from the set $\mathcal{A}$ and call this new set $\mathcal{A}_1$, then we can repeat the steps performed above but now working with the new sets of vectors: $\mathcal{A}_1$ with $N_a - 1$ elements and $\mathcal{B}_1$ with N_b elements.

Our strategy is to repeat the procedure above another $N_b - 1$ times until all of the vectors in $\mathcal{B}$ have been removed from $\mathcal{B}_{N_b}$ leaving $\mathcal{B}_{N_b} = \{a_k\}_{1 \le k \le N_b}$ while $\mathcal{A}_{N_b} == \{a_k\}_{N_b+1 \le k \le N_a}$. At the n^{th} step we will show that the set $\mathcal{B}_n$ is a maximal, linearly independent set. We will then reach a contradiction that the set $\mathcal{B}_{N_b}$ which contains only the first N_b of the N_a vectors a_i forms a basis while we know that it is only the larger set of $N_a > N_b$ vectors $\mathcal{A}$ which is a basis. We will then have shown that $N_a > N_b$ is false or $N_a \le N_b$ as we had intended.

Thus, the final step in our proof is to demonstrate that we can carry out the n^{th} inductive step above showing that if $\mathcal{B}_{n-1}$ is a basis then we can add a_n to this set, remove one of the remaining b_k from that set and be left with the new set $\mathcal{B}_n$ which remains a basis. To do this we need to review the three steps above taken to show that starting with the basis $\mathcal{B}$ the new set $\mathcal{B}_1$ was also a basis and show that these same steps can be repeated to show that if $\mathcal{B}_{n-1}$ is a basis then $\mathcal{B}_n$ must be as well.

Step #1 can be repeated easily in this context. Because $\mathcal{B}_{n-1}$ is a basis, when we add the vector a_n and the new set of $N_b + 1$ vectors must be linearly dependent so the analog of Eq. (1) must hold:

$$\widetilde{c}_1 a_1 + \widetilde{c}_2 a_2 \ldots + \widetilde{c}_n a_n + c_1 b_1 + c_2 b_2, \ldots, c_{N_b-(n-1)} b_{N_b-(n-1)} = 0. \tag{7}$$

As in our earlier argument, the set $\{b_1, b_1, b_{N_b-(n-1)}\}$ is linearly independent so one or more of the coefficients $\{\widetilde{c}_k\}_{1\le k\le n}$ must be non-zero which in turn requires one or more of the coefficients $\{c_k\}_{1\le k\le N_b-(n-1)}$ to be non-zero. As before we will relabel the vectors $\{b_k\}_{1\le k\le N_b-(n-1)}$ so that those with non-zero coefficients will have the largest values of the index k and then, using these new labels, drop the vector $b_{N_b-(n-1)}$ to define the set of N_b vectors:

$$\mathcal{B}_n = \{a_1, a_2, \dots a_n, b_1, b_2, \dots, b_{N_b-n}\}, \tag{8}$$

constructed in close analogy with Eq. (2).

Next we repeat step #2 to show that the set $\mathcal{B}_n$ is linearly independent. As before we assume that the set $\mathcal{B}_n$ is linearly dependent and identify coefficients $\widetilde{d}_i$ and d_j, some of which are non-zero, which satisfy:

$$\widetilde{d}_1 a_1 + \widetilde{d}_2 a_2 \cdots + \widetilde{d}_n a_n + d_1 b_1 + d_1 b_1 \dots + d_{N_b-n} b_{N_b-n} = 0. \tag{9}$$

Since the set B_{n-1} is linearly independent, the extra vector a_n appearing in Eq. (9) must have a non-zero coefficient, allowing us to express a_n in terms of the other vectors in Eq. (9):

$$a_n = -\frac{1}{\widetilde{d}_n}\left\{\widetilde{d}_1 a_1 + \widetilde{d}_2 a_2 \cdots + \widetilde{d}_{n-1} a_{n-1} + d_1 b_1 + d_1 b_1 \dots + d_{N_b-n} b_{N_b-n}\right\}. \tag{10}$$

Following the earlier step #2, we then substitute the expression for a_n into Eq. (7) to obtain the relation:

$$\sum_{i=1}^{n-1} \widetilde{c}_i a_i - \frac{\widetilde{c}_n}{\widetilde{d}_n}\left\{\sum_{j=1}^{n-1} \widetilde{d}_j a_j + \sum_{l=1}^{N_b-n} d_l b_l\right\} + \sum_{k=1}^{N_b-(n-1)} c_k b_k = 0. \tag{11}$$

Since the vectors in Eq. (11) all belong to the set $\mathcal{B}_n$ while the vector $b_{N_b-(n-1)}$ appears in only one term and its coefficient $c_{N_b-(n-1)}$ is by construction non-zero, Eq. (11) is a linear relation between the vectors in $\mathcal{B}_{n-1}$ with non-zero coefficients which violates the linear independence of the set $\mathcal{B}_{n-1}$. Thus, our hypothesis that the set $\mathcal{B}_n$ is linearly dependent must be false.

Our final step is the analogue of step #3 above which shows that $\mathcal{B}_n$ is also maximal. As in that earlier step we begin with an arbitrary vector $w \in V$ and write in in terms of the basis $\mathcal{B}_{n-1}$:

$$w = \sum_{i=1}^{n-1} \widetilde{f}_i a_i + \sum_{i=1}^{N_b-(n-1)} f_i b_i. \tag{12}$$

We then use Eq. (7) to express the vector $b_{N_b-(n-1)}$ in terms of the vectors in $\mathcal{B}_n$.

$$b_{N_b-(n-1)} = -\frac{1}{c_{N_b-(n-1)}}\left\{\widetilde{c}_1 a_1 + \widetilde{c}_2 a_2 \dots + \widetilde{c}_n a_n + c_1 b_1 + c_2 b_2, \dots, c_{N_b-n} b_{N_b-n}\right\}. \tag{13}$$

When this expression is substituted into Eq. (12), this equation then expresses a general vector w in terms of the vectors in $\mathcal{B}_n$ showing that $\mathcal{B}_n$ is indeed a basis too. This completes our inductive proof.