Can we solve for $x(t)$ if we know

$$\frac{d^2x}{dt^2} = \frac{F(t)}{m} \; ?$$

Yes! If we are given $x(0)$ and $\frac{dx}{dt}(0)$

Write Newton's law as two coupled $1^{st}$ order equations

$$\frac{dx}{dt} = v(t) \quad \& \quad \frac{dv}{dt} = \frac{F(t)}{m}$$

Divide time into N small intervals of length $\Delta t$ and use

$$x(\Delta t) = x(0) + v(0)\,\Delta t + O(\Delta t^2)$$

$$v(\Delta t) = v(0) + \frac{F(0)}{m}\Delta t + O(\Delta t^2)$$

Each is an application of a Taylor series:

$$f(t+\Delta t) = f(t) + \frac{df}{dt}(t)\,\Delta t + \frac{1}{2}\frac{d^2f}{dt^2}(\Delta t)^3 + \frac{1}{3!}\frac{d^3f}{dt^3}(\Delta t)^3 + \cdots$$



Can repeat to find $x(n\Delta t)$ & $v(n\Delta t)$

$$x(n\Delta t) = x((n-1)\Delta t) + v((n-1)\Delta t)\,\Delta t$$

$$v(n\Delta t) = v((n-1)\Delta t) + \frac{F((n-1)\Delta t)}{m}\Delta t$$

Error in $x(n\Delta t)$ or $v(n\Delta t)$ will be $(\Delta t)^2$ error at each step $\times$ $n$ steps

$$\text{error} \sim (\Delta t)^2 \times n = (n\,\Delta t)\,\Delta t = t \cdot \Delta t$$


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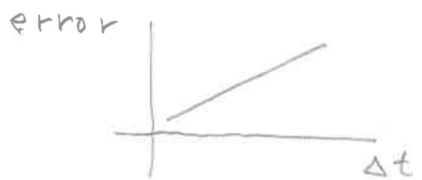


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For two special cases:

$$F(x,t) = F(t) \quad \text{and} \quad F(x,t) = F(x)$$

Newton's equation can be solved by standard integration

1. $m\dfrac{d^2x}{dt^2} = F(t)$, integrate both sides with respect to $t$:

$$\frac{dx}{dt}(t) = \frac{dx}{dt}(0) + \int_0^t \frac{F(t')}{m}\,dt'$$

Then integrate w.r.t. $t$ again

$$x(t) = x(0) + \frac{dx}{dt}(0)\,t + \int_0^t dt'' \int_0^{t''} \frac{F(t')}{m}\,dt'$$

2. $m\dfrac{d^2x}{dt^2} = F(x(t))$, define $U(x) = -\displaystyle\int_0^x F(x')\,dx'$

prove (later) $E = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + U(x)$ is a constant (energy conservation)

$$t = \int_0^t dt' = \int_{x(0)}^{x(t)} \frac{dx}{v(x)} = \int_{x(0)}^{x(t)} \frac{dx}{\sqrt{\frac{2}{m}\left(E - U(x)\right)}}$$

Will be covered later ....

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1. Kinematics - describe the motion of this particle in 3 dimensions

Choose a fixed point P and locate m by a "displacement" $\vec{D}$, an arrow with head at m and tail at P,


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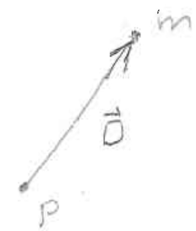


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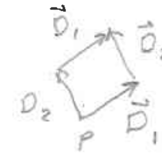
① Addition




$\vec{D}_1 + \vec{D}_2 = \vec{D}_2 + \vec{D}_1$

② There is a zero displacement $\vec{0}$  $\vec{0} + \vec{D} = \vec{D}$

③ Additive inverse: given $\vec{D}$ can find $-\vec{D}$ obeying $\vec{D} + (-\vec{D}) = \vec{0}$ (reverse direction of arrow)

④ Scalar multiplication: given a real number r, define $r\vec{D}$ as a displacement of length $|r|\,|\vec{D}|$ (where $|\vec{D}|$ is the length of $\vec{D}$). in the direction of $\vec{D}$ if $r \geq 0$ or $-\vec{D}$ if $r < 0$. ⓐ $r_1(r_2\vec{D}) = (r_1 r_2)\vec{D}$

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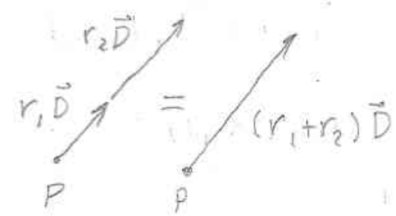
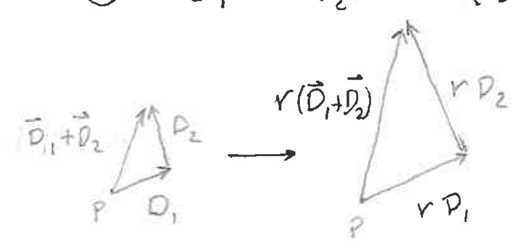
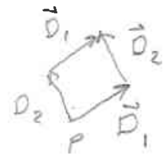
ⓒ $r\vec{D}_1 + r\vec{D}_2 = r(\vec{D}_1 + \vec{D}_2)$



① Addition  $\vec{D}_1 + \vec{D}_2 = \vec{D}_2 + \vec{D}_1$

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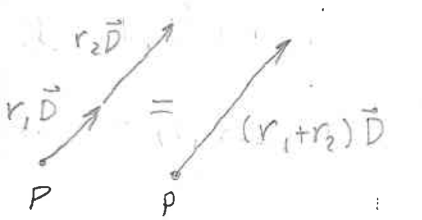
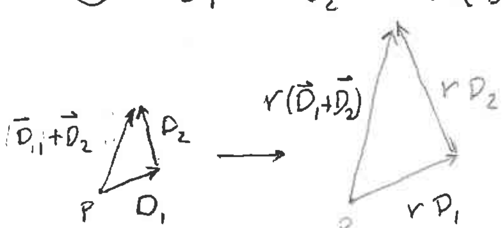




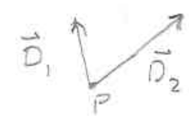

A set of quantities with these operations obeying these conditions is called a vector space and its elements are called vectors.

Important concepts from linear algebra:

① A set of $N$ vectors $\vec{U}_1, \vec{U}_2, \ldots \vec{U}_N$ is linearly independent if the only set of real numbers which obey

$$r_1\vec{U}_1 + r_2\vec{U}_2 + \cdots + r_N\vec{U}_N = 0$$

is the single choice $r_i = 0$, $1 \leq i \leq N$



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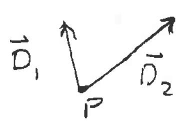


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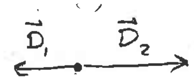


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Important facts

- Given a basis $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_N\}$ for our vector space V, any vector $\vec{\omega} \in V$ can be written as a linear combination of the basis:

proof: by definition, there must exist $r, r_1, r_2, \dots r_N$ which obey

$$r\vec{\omega} + r_1 \vec{v}_1 + r_2 \vec{v}_2 + \dots + r_N \vec{v}_N = 0$$

and $r$ cannot be zero. Then

$$\vec{\omega} = -\left(\frac{r_1}{r}\right)\vec{v}_1 - \left(\frac{r_2}{r}\right)\vec{v}_2 \cdots - \left(\frac{r_N}{r}\right)\vec{v}_N$$

We can also show that these coefficients are unique:

If $\vec{\omega} = x_1 \vec{v}_1 + x_2 \vec{v}_2 + \dots + x_N v_N$

and $\vec{\omega} = y_1 \vec{v}_1 + y_2 \vec{v}_2 + \dots + y_N v_N$

we can subtract the two equation

$$0 = (x_1 - y_1)\vec{v}_1 + (x_2 - y_2)\vec{v}_2 + \dots + (x_N - y_N)\vec{v}_N$$

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Important facts

- Given a basis $\{\vec{v}_1, \vec{v}_2, \cdots \vec{v}_N\}$ for our vector space $V$, any vector $\vec{w} \in V$ can be written as a linear combination of the basis:

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proof: let $A = \{\vec{a}_1, \cdots \vec{a}_N\}$ and $B = \{\vec{b}_1, \ldots \vec{b}_{N'}\}$ be two sets of basis vectors. Consider $B_1 = \{\vec{a}_1, \vec{b}_1, \cdots, \vec{b}_{N'}\}$ which must be linearly dependent. Find the smallest $j$ with $\{\vec{a}_1, \vec{b}_1, \cdots, \vec{b}_j\}$ dependent. Remove $b_j$ and call the new set $B_1'$, also a basis. Then consider $B_2 = \{a_2, B_1'\}$ and repeat the procedure. If $N' < N$ we will remove all of the $\vec{b}_j$'s before we have included all the $\vec{a}_i$'s and claim $B_{N'}' = \{a_{N'}, a_{N'-1}, \ldots a_1\}$ is a basis. $\Rightarrow N' \geq N$ then reverse $A \not\!\!+ B$

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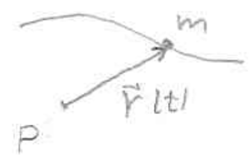
We can use displacement vectors to locate a point partice in 3-dim




Vectors permit all of operations need to write down Newton's Law

$$\vec{v}(t) = \frac{d\vec{r}}{dt}(t) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\cdot[\vec{r}(t+\Delta t)-\vec{r}(t)]$$

scalar multiplication

additive inverse and addition

$$\vec{a}(t)=\frac{d\vec{v}}{dt}(t)$$

Velocity and acceleration are both vectors and $\vec{v}=0$ & $\vec{a}=0$ are less arbitrary than $\vec{r}=0$

2. Dynamics is now obvious

$$\underbrace{F=ma}_{1\text{-dim}} \longrightarrow \underbrace{\vec{F}=m\vec{a}}_{3\text{-dim}}$$

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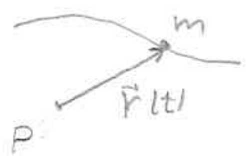


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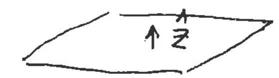
Return to particle of mass m moving near the earth's surface




introduce upward vector $\hat z$ of unit length

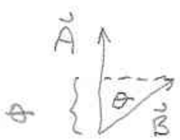
Let $\vec F_{grav} = -mg\,\hat z$

$$m\frac{d^2\vec r}{dt^2} = -mg\,\hat z \quad \text{solved}$$

$$\frac{d\vec r}{dt} = -gt\,\hat z + \vec v(0) \quad \text{and}$$

$$\vec r(t) = -\tfrac{1}{2}g t^2\,\hat z + \vec v(0)\,t + \vec r(0)$$

3. Projection and dot products




$\underbrace{|\vec B|\cos\theta}$ projection of $\vec B$ and the direction of $\vec A$

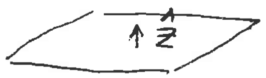
Define

$$(\vec A, \vec B) = |A|\,|B|\cos\theta$$

adding $|\vec A|$ makes the product symmetric

$$(\vec A, \vec B) = (\vec B, \vec A)$$

Return to particle of mass $m$ moving near the earth's surface

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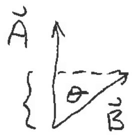



projection of $\vec{B}$ and the direction of $\vec{A}$

Define

$$(\vec{A}, \vec{B}) = |A||B|\cos\theta$$

adding $|\vec{A}|$ makes the product symmetric

$$(\vec{A}, \vec{B}) = (\vec{B}, \vec{A})$$



$(\vec{A}, \vec{B}) \equiv \vec{A}\cdot\vec{B}$ has three important properties:

① $(A, A) = |A|^2\cos(0) = |A|^2 \geq 0$

$\quad$ + $\vec{A}\cdot\vec{A} = 0 \Rightarrow \vec{A} = \vec{0}$

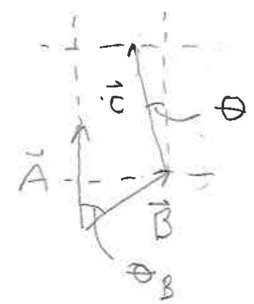
② $(\vec{A}, r\vec{B}) = |\vec{A}||r\vec{B}|\cos\theta = r|A||B|\cos\theta$

$\quad = r(\vec{A}, \vec{B})$

[Linear under scalar multiplication]

③ $(\vec{A}, \vec{B}+\vec{C}) = (\vec{A}, \vec{B}) + (\vec{A}, \vec{C})$ $\quad$ proof:

2-dim




projections of $\vec{B}$ and $\vec{C}$ onto the direction of $\vec{A}$ simply add

3-dim