$(\vec{A}, \vec{B}) \equiv \vec{A} \cdot \vec{B}$ has three important properties:

① $(A, A) = |A|^2 \cos(0) = |A|^2 \geq 0$

$+ \ \vec{A} \cdot \vec{A} = 0 \Rightarrow \vec{A} = \vec{0}$

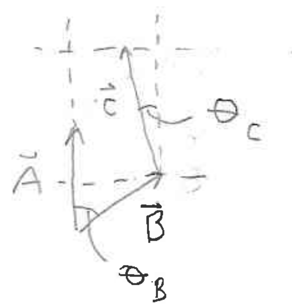
② $(\vec{A}, r\vec{B}) = |\vec{A}||r\vec{B}| \cos\theta = r|A||B|\cos\theta$

$= r(\vec{A}, \vec{B})$

[Linear under scalar multiplication]

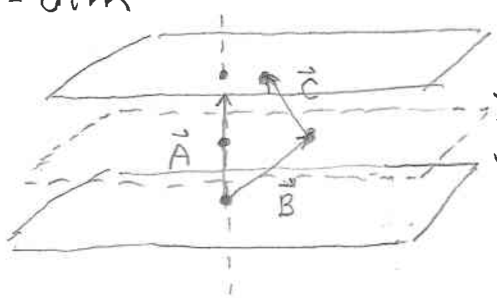
③ $(\vec{A}, \vec{B}+\vec{C}) = (\vec{A}, \vec{B}) + (\vec{A}, \vec{C})$ proof:

2-dim



projections of $\vec{B}$ and $\vec{C}$ onto the direction of $\vec{A}$ simply add

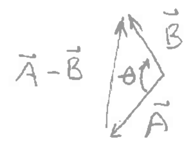
3-dim



$|\vec{C}| \cos\theta_C$

$|\vec{B}| \cos\theta_B$

Easy proof of law of cosines



$$|\vec{A}-\vec{B}|^2 = (\vec{A}-\vec{B}, \vec{A}-\vec{B})$$

$$= (\vec{A}, \vec{A}) + (\vec{B}, \vec{B}) - 2(\vec{A} \cdot \vec{B})$$

$$= |A|^2 + |\vec{B}|^2 - 2|A||B|\cos\theta$$

Use dot product to define a special basis of orthogonal, unit-length vectors: $\hat{e}_1, \hat{e}_2, \hat{e}_3$

$$\hat{e}_1 \cdot \hat{e}_1 = \hat{e}_2 \cdot \hat{e}_2 = \hat{e}_3 \cdot \hat{e}_3 = 1$$

$$\hat{e}_1 \cdot \hat{e}_2 = \hat{e}_1 \cdot \hat{e}_3 = \hat{e}_2 \cdot \hat{e}_3 = 0$$

Sine they form a basis, for any vector $\vec{A}$:

$$\vec{A} = a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3$$

Because the $\{\hat{e}_i\}$ are orthonormal we can easily recover the coefficients a_i

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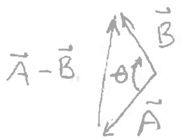

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$$\hat{e}_1\cdot\vec{A} = \hat{e}_1\cdot[a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3]$$
$$= a_1\,\hat{e}_1\cdot\hat{e}_1 = a_1$$

Thus $a_i = \hat{e}_i\cdot\vec{A}$ & $\vec{A} = \sum_{i=1}^{3}(\vec{A}\cdot\hat{e}_i)\,\hat{e}_i$

This allows us to become more quantitative and replace our notion of a vector with its three coordinates

$$\vec{A} \longleftrightarrow (a_1, a_2, a_3)$$

all of our abstract vector operations become explicit if written in terms of coordinate:

$$r\vec{A} = r(a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3)$$
$$= (ra_1)\hat{e}_1 + (ra_2)\hat{e}_2 + (ra_3\hat{e}_3)$$

$$r\vec{A} \longleftrightarrow (ra_1, ra_2, ra_3)$$

$$\hat{e}_1 \cdot \vec{A} = \hat{e}_1 \cdot [a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3]$$

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$$r\vec{A} \longleftrightarrow (ra_1, ra_2, ra_3)$$

$$\vec{A} + \vec{B} = (a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3) + (b_1 \hat{e}_1 + b_2 \hat{e}_2 + b_3 \hat{e}_3)$$

$$= (a_1 + b_1)\hat{e}_1 + (a_2 + b_2)\hat{e}_2 + (a_3 + b_3)\hat{e}_3$$

$$\vec{A} + \vec{B} \longleftrightarrow (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

$$\vec{A} \cdot \vec{B} = (a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3) \cdot (b_1 \hat{e}_1 + b_2 \hat{e}_2 + b_3 \hat{e}_3)$$

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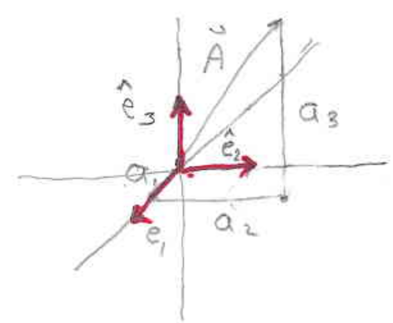
Thus $|A|^2 = a_1^2 + a_2^2 + a_3^2$

Pythagorous theorem in 3-dim



$$\vec{A} = a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3$$

$$|A|^2 = a_1^2 + a_2^2 + a_3^2$$

$$\vec{A}+\vec{B} = (a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3) + (b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3)$$

$$= (a_1+b_1)\hat{e}_1 + (a_2+b_2)\hat{e}_2 + (a_3+b_3)\hat{e}_3$$

$$\vec{A}+\vec{B} \longleftrightarrow (a_1+b_1,\ a_2+b_2,\ a_3+b_3)$$

$$\vec{A}\cdot\vec{B} = (a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3)\cdot(b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3)$$

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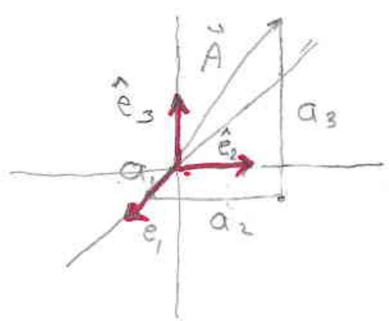
Pythagorous theorem in 3-dim



$$\vec{A} = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3$$

$$|A|^2 = a_1^2 + a_2^2 + a_3^2$$

4. We can use coordinates to make Newton's 2^{nd} law explicit in 3-dim

$$\hat{e}_i \left\{ m\frac{d^2\vec{r}}{dt^2} = \vec{F} \right\}$$

$$m\frac{d}{dt}\underbrace{(\hat{e}_i\cdot\vec{r})}_{r_i} = \underbrace{\hat{e}_i\cdot\vec{F}}_{F_i} \qquad 1\le i\le 3$$

So we have 3 explicit equations

$$m\frac{d^2 r_1}{dt^2} = F_1(r_1, r_2, r_3, t)$$

$$m\frac{d^2 r_2}{dt^2} = F_2(r_1, r_2, r_3, t)$$

$$m\frac{d^2 r_3}{dt^2} = F_3(r_1, r_2, r_3, t)$$

5. Two examples

a) Projectile motion

$$m\frac{d^2\vec{r}}{dt^2} = -mg\,\hat{e}_3$$

$$\vec{r}(t) = \vec{r}(0) + \vec{v}(0)t - \tfrac{1}{2}mgt^2\hat{e}_3$$

Choose $\hat{e}_1$ so $\vec{v}(0)$ lies in $\hat{e}_1 - \hat{e}_3$ plane

4. We can use coordinates to make Newton's 2^{nd} law explicit in 3-dim

$$\hat{e}_i \left\{ m \frac{d^2\vec{r}}{dt^2} = \vec{F} \right\}$$

$$m \frac{d}{dt}(\underbrace{\hat{e}_i \cdot \vec{r}}_{r_i}) = \underbrace{\hat{e}_i \cdot \vec{F}}_{F_i} \qquad 1 \le i \le 3$$

So we have 3 explicit equations

$$m \frac{d^2 r_1}{dt^2} = F_1(r_1, r_2, r_3, t)$$

$$m \frac{d^2 r_2}{dt^2} = F_2(r_1, r_2, r_3, t)$$

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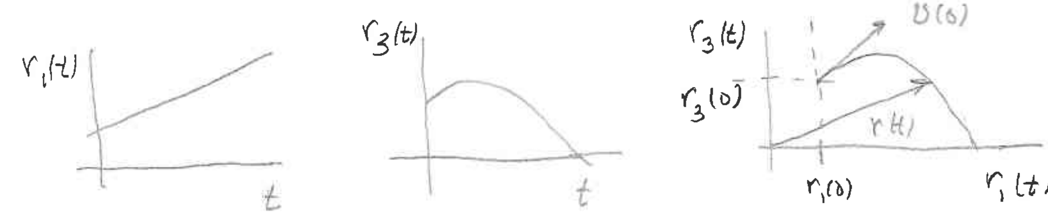
Choose $\hat{e}_1$ so $\vec{v}(0)$ lies in $\hat{e}_1 - \hat{e}_3$ plane

$$r_1(t) = r_1(0) + v_1(0)t$$

$$r_2(t) = r_2(0)$$

$$r_3(t) = r_3(0) + v_3(0)t - \frac{1}{2}gt^2$$

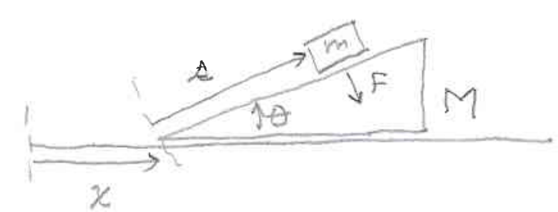


$$\vec{v}(0) = v_1(0)\hat{e}_1 + v_3(0)\hat{e}_3$$

tangent to $\vec{r}(t)$ at $t=0$

$$\left.\frac{\Delta r_3(t)}{\Delta r_1(t)}\right|_{t=0} = \frac{v_3(0)\Delta t}{v_1(0)\Delta t} = \frac{v_3(0)}{v_1(0)}$$

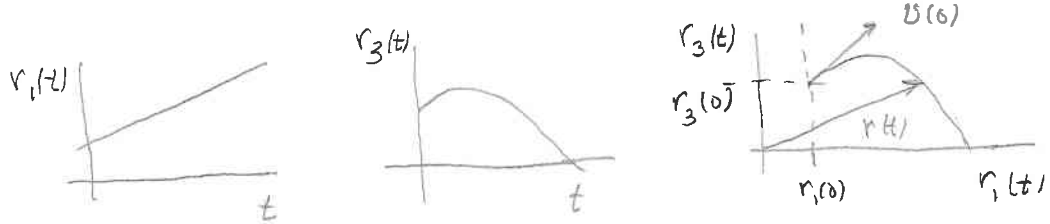
b) A block of mass m slides on a wedge of mass M which slides on a horizontal table. If both move without friction find the acceleration of the wedge



$$r_1(t) = r_1(0) + v_1(0)\, t$$

$$r_2(t) = r_2(0)$$

$$r_3(t) = r_3(0) + v_3(0)\, t - \frac{1}{2} g t^2$$



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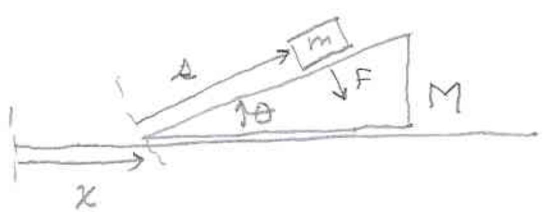
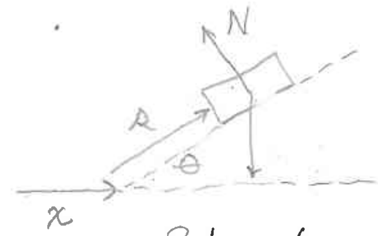
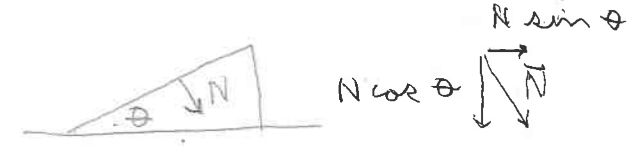




Block



Wedge

Veritical forces and acceleration:

$$m \ddot{s} \sin\theta = N\cos\theta - mg$$

Horizontal

$$-m(\ddot{x} + \ddot{s}\cos\theta) = -N\sin\theta$$

$\times \sin\theta$

$\times (-\cos\theta)$

Horizontal forces and acceleration:

$$M\ddot{x} = N\sin\theta$$

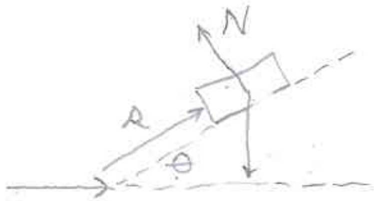
3 equations, 3 unknowns

$$m\sin\theta\, \ddot{x} = -N + mg\cos\theta$$

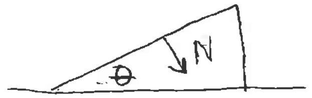
$$M\ddot{x}\frac{1}{\sin\theta}$$

$$(m\sin^2\theta + M)\,\ddot{x} = mg\sin\theta\cos\theta$$

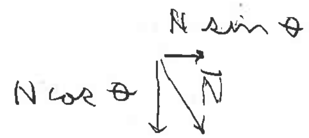
$$\ddot{x} = \frac{m\sin\theta\cos\theta}{m\sin^2\theta + M}\, g$$



Block



Wedge



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$$m\ddot{s}\sin\theta = N\cos\theta - mg$$

Horizontal forces: and acceleration:

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3 equations, 3 unknowns

$\times\sin\theta$

$\times(-\cos\theta)$

$$m\sin\theta\,\ddot{x} = -N + mg\cos\theta$$

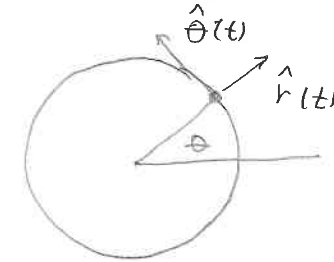
$$M\ddot{x}\,\frac{1}{\sin\theta}$$

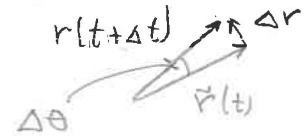
$$(m\sin^2\theta + M)\ddot{x} = mg\sin\theta\cos\theta$$

$$\ddot{x} = \frac{m\sin\theta\cos\theta}{m\sin^2\theta + M}\,g$$

6. Circular motion – two ways

a) abstract vectors



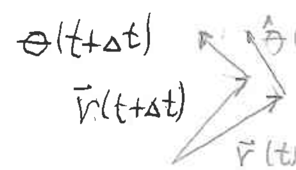


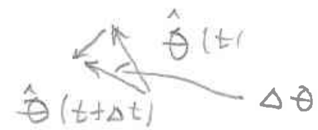
$$\Delta\vec{r} = \hat{\theta}\cdot r\cdot\underbrace{\Delta\theta}_{\dot{\theta}\,\Delta t}$$

$$\vec{v}(t) = \frac{\Delta\vec{r}}{\Delta t} = \hat{\theta}\,r\,\dot{\theta} = \hat{\theta}\,r\,\omega(t) \quad \text{if } \dot{\theta} = \omega$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left\{\hat{\theta}(t)\,r\,\omega(t)\right\}$$

$$= \frac{d\hat{\theta}}{dt}\,r\,\omega(t) + \hat{\theta}\,r\,\dot{\omega}(t)$$





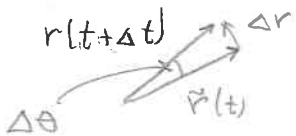
$$\frac{d\hat{\theta}}{dt} = -\hat{r}\,\frac{\Delta\theta}{\Delta t} = -\hat{r}\,\omega$$

$$\therefore\ \vec{a} = \underbrace{-\vec{r}\,\omega^2}_{\text{centripetal acceleration}} + r\,\hat{\theta}\,\dot{\omega}$$

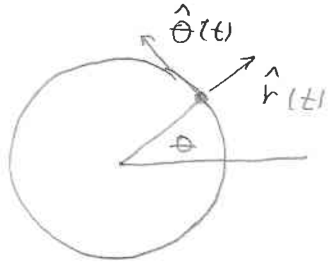
Particle traveling in a circle must be pulled inward by a centripetal force $\vec{F} = -\vec{r}\,\omega^2 m$

6. Circular motion – two ways

a) abstract vectors



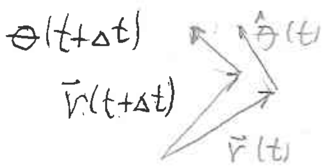
$\Delta\vec r = \hat\theta \cdot r \cdot \underbrace{\Delta\theta}_{\dot\theta\,\Delta t}$

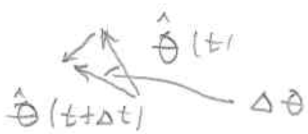


$\vec v(t) = \frac{\Delta\vec r}{\Delta t} = \hat\theta\, r\,\dot\theta = \hat\theta\, r\,\omega(t)$ if $\dot\theta = \omega$

$\vec a(t) = \frac{d\vec v}{dt} = \frac{d}{dt}\{\hat\theta(t)\, r\, \omega(t)\}$

$= \frac{d\hat\theta}{dt}\, r\,\omega(t) + \hat\theta\, r\,\dot\omega(t)$





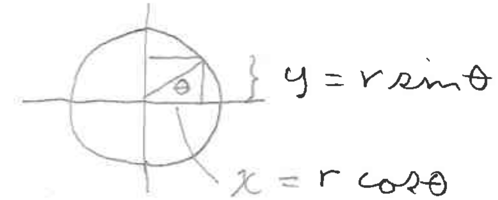
$\frac{d\hat\theta}{dt} = -\hat r\,\frac{\Delta\theta}{\Delta t} = -\hat r\,\omega$

& $\vec a = \underbrace{-\vec r\,\omega^2}_{\text{centripetal acceleration}} + r\,\hat\theta\,\dot\omega$

Particle traveling in a circle must be pulled inward by a centripetal force $\vec F = -\vec r\,\omega^2 m$

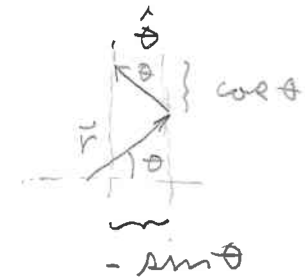
b) coordinates

$\vec v = \frac{d}{dt}(r\cos\theta,\ r\sin\theta)$



$= \left(+ r\sin\theta\,\frac{d\theta}{dt},\ r\cos\theta\,\frac{d\theta}{dt}\right)$

$= r\,\frac{d\theta}{dt}\,\underbrace{(-\sin\theta,\ \cos\theta)}_{\hat\theta}$

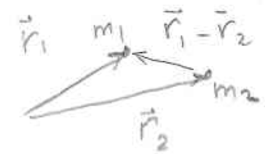


$\vec a = \frac{dv}{dt} = \frac{d}{dt}\, r\,\omega(t)\,(-\sin\theta(t),\ \cos\theta(t))$

$= r\,\dot\omega\,\hat\theta + r\,\omega^2(t)\,(-\cos\theta(t),\ -\sin\theta(t))$

$= r\,\dot\omega\,\hat\theta - \vec r\,\omega^2$ ✓

7. Newton's law of gravitation



$\vec F_{1\ \text{on}\ 2} = \frac{G m_1 m_2}{|r_1 - r_2|^2}\,\frac{\vec r_1 - \vec r_2}{|\vec r_1 - \vec r_2|}$

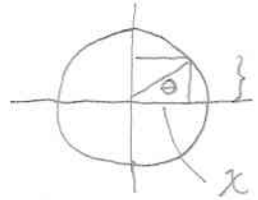
$G = 6.67\times10^{-8}\ \frac{\text{dyne}\cdot\text{cm}^2}{\text{gm}}$

recall $1\ \text{Newton} = \frac{\text{kg m}}{\text{sec}^2}$

$1\ \text{dyne} = \frac{\text{g}\cdot\text{cm}}{\text{sec}^2} = 10^{-5}\ \text{Newton}$

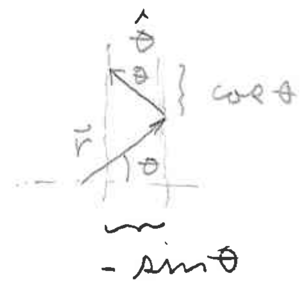
b) coordinates

$$\vec{v} = \frac{d}{dt}(r\cos\theta,\ r\sin\theta)$$



$$= \left(-r\sin\theta\,\frac{d\theta}{dt},\ r\cos\theta\,\frac{d\theta}{dt}\right)$$

$$= r\frac{d\theta}{dt}\underbrace{(-\sin\theta,\ \cos\theta)}_{\hat{\theta}}$$

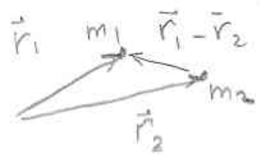


$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\, r\,\omega(t)\left(-\sin\theta(t),\ \cos\theta(t)\right)$$

$$= r\dot{\omega}\,\hat{\theta} + r\omega^2(t)\left(-\cos\theta(t),\ -\sin\theta(t)\right)$$

$$= r\dot{\omega}\,\hat{\theta} - \hat{r}\,\omega^2 \checkmark$$

7. Newton's law of gravitation



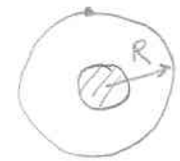
$$\vec{F}_{1\text{ on }2} = \frac{G m_1 m_2}{|r_1 - r_2|^2}\,\frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}$$

$$G = 6.67\times10^{-8}\ \frac{\text{dyne}\cdot\text{cm}^2}{\text{gm}}$$

recall $1\ \text{Newton} = \dfrac{\text{kg m}}{\text{sec}^2}$

$$1\ \text{dyne} = \frac{\text{g}\cdot\text{cm}}{\text{sec}^2} = 10^{-5}\ \text{Newton}$$

Problem: Find the radius of a geosynchrous orbit



$$\omega = \frac{2\pi}{\text{day}}$$

$$\cancel{m_1} R\,\omega^2 = \frac{G\cancel{m_1} M_E}{R^2}$$

$$R^3 = \frac{G M_E}{\omega^2} = \underbrace{\frac{G M_E}{R_E^2}}_{g} \times \frac{R_E^2}{\omega^2}$$

$$= \frac{32\ \text{ft/sec}^2 \times (3960\ \text{mi})^2}{(2\pi)^2}\,(24\times3600\ \text{sec})^2 \times \frac{1\ \text{mi}}{5280\ \text{ft}}$$

$$R = \left[\frac{32\times(3960)^2\,(24\times3600)^2}{(2\pi)^2\ 5280}\right]^{1/3}\text{mi}$$

$$= 26{,}200\ \text{mi}$$

8. Changing basis vectors

$$\vec{A} = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3$$

$$= a_1'\hat{e}_1' + a_2'\hat{e}_2' + a_3'\hat{e}_3'$$

- Change to a more convenient basis
- Check that physical law does not depend on choice of basis

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- Change to a more convenient basis
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Express the new basis in terms of the old one

$$\hat{e}_i' = M_{i1} \hat{e}_1 + M_{i2} \hat{e}_2 + M_{i3} \hat{e}_3 \qquad i = 1, 2 \,\&\, 3$$

note $\hat{e}_i' \cdot \hat{e}_j = M_{ij}$

Then $a_i' = \hat{e}_i' \cdot A = \hat{e}_i' \cdot [a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3]$

$$= M_{i1} a_1 + M_{i2} a_2 + M_{i3} a_3$$

Thus the a_i' are given by matrix multiplication

$$\begin{pmatrix} a_1' \\ a_2' \\ a_3' \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

Recall in general

$$\underset{l \times m}{X} \cdot \underset{m \times k}{Y} = \underset{l \times k}{(XY)}$$

$$(XY)_{ab} = \sum_{\kappa=1}^{m} X_{a\kappa} Y_{\kappa b} \qquad \begin{matrix} 1 \le a \le l \\ 1 \le b \le k \end{matrix}$$

Next change basis twice

$$A = \sum_{i=1}^{3} a_i \hat{e}_i = \sum_{i=1}^{3} a_i' \hat{e}_i' = \sum_{i=1}^{3} a_i'' \hat{e}_i''$$

Express the new basis in terms of the old one

$$\hat{e}'_i = M_{i1}\hat{e}_1 + M_{i2}\hat{e}_2 + M_{i3}\hat{e}_3 \qquad i = 1, 2 \;\&\; 3$$

note $\hat{e}'_i \cdot \hat{e}_j = M_{ij}$

Then $a'_i = \hat{e}'_i \cdot A = \hat{e}'_i \cdot [a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3]$

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Next change basis twice

$$A = \sum_{i=1}^{3} a_i \hat{e}_i = \sum_{i=1}^{3} a'_i \hat{e}'_i = \sum_{i=1}^{3} a''_i \hat{e}''_i$$

Introduce $M_{ij} = \hat{e}'_i \cdot \hat{e}_j \qquad \hat{e}_j \to \hat{e}'_i$

$\& \quad M'_{ij} = \hat{e}''_i \cdot \hat{e}'_j \qquad \hat{e}'_j \to \hat{e}''_i$

Then $a' = Ma \qquad a'' = M'a' = M'(Ma)$

or $a''_i = \sum_{k=1}^{3} M'_{ik} \left(\sum_{j=1}^{3} M_{kj} a_j \right)$

$$= \sum_{j=1}^{3} \underbrace{\left(\sum_{k=1}^{3} M'_{ik} M_{kj} \right)}_{(M'M)_{ij}} a_j$$

same 9 products

and $(M'M)$ transforms $\hat{e}_i$ into $\hat{e}''_j$!

Consider the special case where $\hat{e}''_i = \hat{e}_i$. Then $a = (M'M)\,a$

or $a_i = \sum_j (M'M)_{ij} a_j$

$$\Rightarrow \quad M_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the identity matrix

If $M'M = I$ define $M' = M^{-1}$