Problem: Find the radius of a geosynchrous orbit

$$\omega = \frac{2\pi}{\text{day}}$$

$$\cancel{m_1} R \omega^2 = \frac{G \cancel{m_1} M_E}{R^2}$$

$$R^3 = \frac{G M_E}{\omega^2} = \underbrace{\frac{G M_E}{R_E^2}}_{g} \times \frac{R_E^2}{\omega^2}$$

$$= \frac{32 \text{ ft/sec}^2 \times (3960 \text{ mi})^2}{(2\pi)^2} (24 \times 3600 \text{ sec})^2 \times \frac{1 \text{ mi}}{5280 \text{ ft}}$$

$$R = \left[\frac{32 \times (3960)^2 (24 \times 3600)^2}{(2\pi)^2 \, 5280} \right]^{1/3} \text{mi}$$

$$= 26{,}200 \text{ mi}$$

8. Changing basis vectors

$$\vec{A} = a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3$$

$$= a_1' \hat{e}_1' + a_2' \hat{e}_2' + a_3' \hat{e}_3'$$

- Change to a more convenient basis
- Check that physical law does not depend on choice of basis

Express the new basis in terms of the old one

$$\hat{e}_i' = M_{i1} \hat{e}_1 + M_{i2} \hat{e}_2 + M_{i3} \hat{e}_3 \qquad i = 1, 2 \,\&\, 3$$

note $\hat{e}_i' \cdot \hat{e}_j = M_{ij}$

Then $a_i' = \hat{e}_i' \cdot A = \hat{e}_i' \cdot [a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3]$

$$= M_{i1} a_1 + M_{i2} a_2 + M_{i3} a_3$$

Thus the a_i' are given by matrix multiplication

$$\begin{pmatrix} a_1' \\ a_2' \\ a_3' \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

Recall in general

$$\underset{l \times m}{X} \cdot \underset{m \times k}{Y} = \underset{l \times k}{(XY)}$$

$$(XY)_{ab} = \sum_{\kappa=1}^{m} X_{a\kappa} Y_{\kappa b} \qquad \begin{matrix} 1 \le a \le l \\ 1 \le b \le k \end{matrix}$$

Next change basis twice

$$A = \sum_{i=1}^{3} a_i \hat{e}_i = \sum_{i=1}^{3} a_i' \hat{e}_i' = \sum_{i=1}^{3} a_i'' \hat{e}_i''$$

Express the new basis in terms of the old one

$$\hat{e}'_i = M_{i1}\hat{e}_1 + M_{i2}\hat{e}_2 + M_{i3}\hat{e}_3 \qquad i = 1, 2, 3$$

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Introduce $M_{ij} = \hat{e}'_i \cdot \hat{e}_j \qquad \hat{e}_j \to \hat{e}'_i$

$M'_{ij} = \hat{e}''_i \cdot \hat{e}'_j \qquad \hat{e}'_j \to \hat{e}''_i$

Then $a' = Ma \qquad a'' = M'a' = M'(Ma)$

or $$a''_i = \sum_{k=1}^{3} M'_{ik} \left(\sum_{j=1}^{3} M_{kj} a_j \right)$$

$$= \sum_{j=1}^{3} \underbrace{\left(\sum_{k=1}^{3} M'_{ik} M_{kj} \right)}_{(M'M)_{ij}} a_j$$

same 9 products

and $(M'M)$ transforms $\hat{e}_i$ into $\hat{e}''_j$!

Consider the special case where $\hat{e}''_i = \hat{e}_i$. Then $a = (M'M)\,a$

or $a_i = \sum_j (M'M)_{ij}\, a_j$

$$\Rightarrow \quad M_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the identity matrix

If $M'M = I$ define $M' = M^{-1}$

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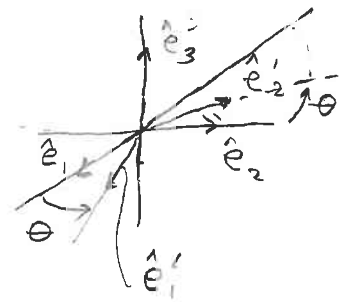
$$(M^t)_{ij} \equiv M_{ji}$$

$M^{-1} = M' = M^t \Rightarrow$ M is an othogonal matrix

Example: rotate basis about $\hat{e}_3$ through θ



$\hat{e}'_1 = \cos\theta\, \hat{e}_1 + \sin\theta\, \hat{e}_2$

$\hat{e}'_2 = -\sin\theta\, \hat{e}_1 + \cos\theta\, \hat{e}_3$

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$$M = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M^t M = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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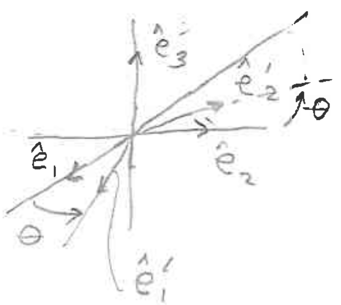

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Physical quantities should change in a prescribed way when the basis is changed

(a) A scalar quantity does not change: t & m are scalars

(b) A vector is define by triples of quantities (V_1, V_2, V_3) which in the new coordinate basis become $V_i' = \sum_{j=1}^{3} M_{ij} V_j$

(c) Nine quantities T_{ij} $\quad 1 \le j \le 3$, $1 \le i \le 3$

define a 2nd rank tensor if

$$T'_{kl} = \sum_{i=1}^{3} \sum_{j=1}^{3} M_{ki} M_{lj} T_{ij}$$

The laws of physics will be the same for all choices of coordinate bases if they equate quantities which transform the same way

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Newton's 2^{nd} Law is a good example

$$F_1 = m \frac{d^2 r_1}{dt^2}$$

$$F_2 = m \frac{d^2 r_2}{dt^2}$$

$$F_3 = m \frac{d^2 r_3}{dt^2}$$

equate vector to vector

$$\sum_{i=1}^{3} M_{ji} \left\{ F_i = m \frac{d^2 r_i}{dt^2} \right\}$$

$$F_j' = m \frac{d^2}{dt^2} \sum_{i=1}^{3} M_{ji} r_i = m \frac{d^2}{dt^2} r_j'$$

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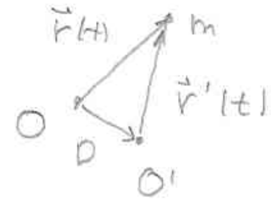
9. Change the origin



$\vec{r}(t) = \vec{r}'(t) + D$

a) If $\vec{D}$ is a constant Newton's law is not changed

$$\vec{F} = m \frac{d^2 \vec{r}}{dt^2} = m \frac{d^2}{dt^2} (\vec{r}'(t) + D) = m \frac{d^2 \vec{r}'}{dt^2}$$

Also true if $\vec{D}(t) = \vec{D}(0) + \vec{v}\, t$

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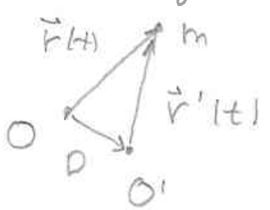

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When $D(t) = D(0) + V t$ (V constant)

$$\vec{v}\,'(t) = \frac{d}{dt}\left(\vec{r}\,'(t)\right) = \frac{d}{dt}\left(\vec{r}(t) - \vec{D}(0) - \vec{V} t\right)$$

$$= v(t) - \vec{V}$$

the velocities change by $-\vec{V}$ but the acceleration is unchanged.

Each choice of origin for which Newton's Law hold defines an inertial reference system. The origins of two such systems must move with a constant relative velocity

If $\frac{d^2 \vec{D}(t)}{dt^2} \neq 0$ Newton's law will be changed

$$m \frac{d^2 \vec{r}\,'}{dt^2} = m \frac{d^2}{dt^2}(\vec{r}(t) - D(t)) = m \frac{d^2 \vec{r}}{dt^2} - m \frac{d^2}{dt^2} D(t)$$

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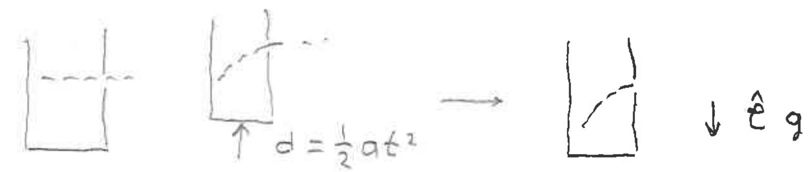
Now we can state Einstein's principle of equivalence: a uniform gravitional force and the fictitious force arising from constant accelergtion cannot be distinguished

$$m\frac{d^2\vec{r}^2}{dt^2} = \vec{F}(t) - m_G g\,\hat{e}$$

$$= \vec{F}(t) - m_I\vec{a}$$

$\Rightarrow$ gravity bends light



Often allows an easy solution

Consider a railroad car on which a pendulum is mounted. Initially the car and pendulum are at rest. If the car is then given a constant acceleration, how high does the pendulum swing?

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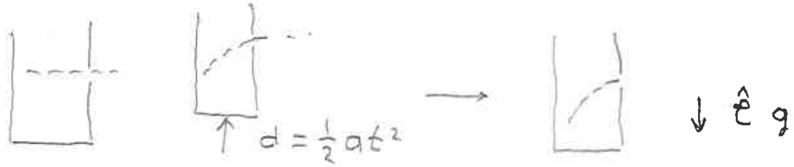

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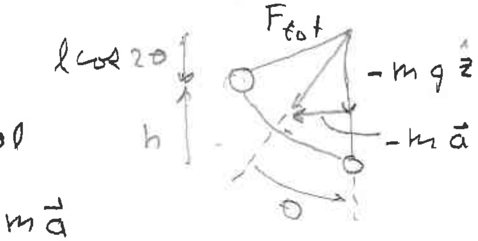
Thus the pendulum swings as if there were a gravitational force $\vec{F}_{tot} = -mg\hat{z} - m\vec{a}$



Swings to height $h = l(1-\cos 2\theta)$

$$\tan\theta = \frac{a}{g}$$

C Momentum

While classical mechanics is entirely determined by Newton's laws, their solution is often made simpler by additional concepts. Momentum is the first of these

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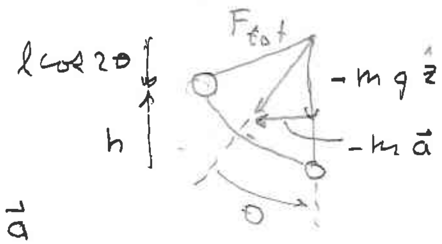

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1: Consider N particles with masses m_i at positions $\vec{r}_i$, $1 \le i \le N$. Assume that particle i acts on particle j with the force $\vec{F}_{i\,on\,j}$

Then for $1 \le i \le N$

$$m_i \frac{d^2\vec{r}_i}{dt^2} = \vec{F}_i^{ext} + \sum_{j \ne i} \vec{F}_{j\,on\,i}$$

Next add all N equations together

$$\sum_{i=1}^{N} m_i \frac{d^2\vec{r}_i}{dt^2} = \underbrace{\sum_{i=1}^{N} \vec{F}_i^{ext}}_{F^{ext}_{tot}} + \underbrace{\sum_{\substack{i,j \\ i \ne j}} \vec{F}_{j\,on\,i}}$$

each pair (i,j) appears twice $\vec{F}_{i\,on\,j} + \vec{F}_{j\,on\,i} = 0$ (Newton's 3rd law)

or

$$\frac{d}{dt}\left[\sum_{i=1}^{N} m_i \frac{d\vec{r}_i}{dt}\right] = \vec{F}^{ext}_{total}$$

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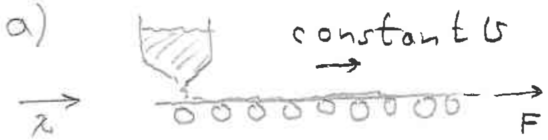
and the total momentum is <u>conserved</u>

Two examples

a)



Sand falls on the belt at the rate λ (gm/sec) what force F must be applied to the belt to keep it moving at a constant velocity v? Total horizontal momentum of all the sand

$$P_x^{tot}(t) = M_{on\,belt}(t)\, v + (M_{hopper} + M_{falling}) \times 0$$

$$\frac{dP_x^{tot}}{dt}(t) = \frac{d}{dt}\left[M_{on\,belt}(t)\right] v = \lambda v$$

$$= F \quad \Rightarrow \quad F = \lambda v$$

Important to identify a single collection of particles (all sand) and determine momentum of all of it!

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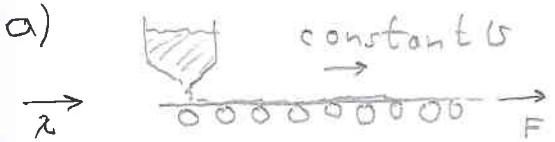

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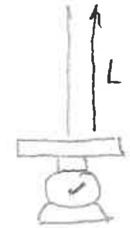
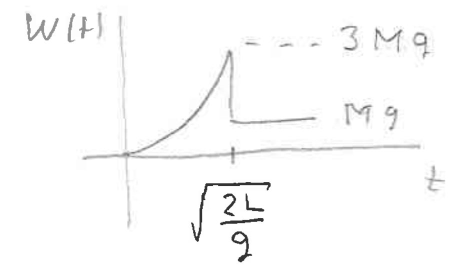


A chain of mass M and length L is hanging at rest just touching the platform of a scale. If the chain is released at t=0, what weight does the scale read as a function of time, W(t)?

$$P_z(t) = \frac{M}{L} \times \underbrace{\left(L - \tfrac{1}{2} g t^2\right)}_{\text{length of moving chain}} \underbrace{gt}_{\text{velocity of moving chain}}$$

$$\frac{dP_z}{dt} = Mg - W(t)$$

$$Mg - \frac{3}{2}\frac{M}{L} g^2 t^2 = Mg - W(t)$$



$$W(t) = \begin{cases} \frac{3}{2}\frac{M}{L} g^2 t^2 & \frac{1}{2} g t^2 < L \\ Mg & \frac{1}{2} g t^2 > L \end{cases}$$

$$W\left(t = \sqrt{\frac{2L}{g}}\right) = 3Mg$$