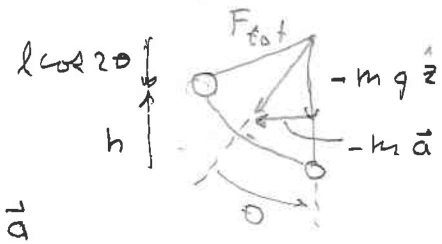
In the non-inertial reference system of the accelerating car the is a fictitious force $\vec{F}_f = -m\vec{a}$

Thus the pendulum swings as if there were a gravitational force $\vec{F}_{tot} = -mg\hat{z} - m\vec{a}$



Swings to height $h = l(1-\cos 2\theta)$

$$\sin\theta = \frac{a}{g}$$

C Momentum

While classical mechanics is entirely determined by Newton's laws, their solution is often made simpler by additional concepts. Momentum is the first of these

1. Consider N particles with masses m_i at positions $\vec{r}_i$, $1 \le i \le N$. Assume that particle i acts on particle j with the force $\vec{F}_{i\,on\,j}$

Then for $1 \le i \le N$

$$m_i \frac{d^2\vec{r}_i}{dt^2} = \vec{F}_i^{\,ext} + \sum_{j \ne i} \vec{F}_{j\,on\,i}$$

Next add all N equations together

$$\sum_{i=1}^{N} m_i \frac{d^2\vec{r}_i}{dt^2} = \underbrace{\sum_{i=1}^{N} F_i^{ext}}_{F^{ext}_{tot}} + \underbrace{\sum_{\substack{i,j \\ i\ne j}} \vec{F}_{j\,on\,i}}$$

each pair (i,j) appears twice $\vec{F}_{i\,on\,j} + \vec{F}_{j\,on\,i} = 0$ (Newton's 3rd law)

or

$$\frac{d}{dt}\left[\sum_{i=1}^{N} m_i \frac{d\vec{r}_i}{dt}\right] = \vec{F}^{\,ext}_{total}$$

$\vec{p}_i = m\dfrac{d\vec{r}_i}{dt} = m\vec{v}_i$ momentum of particle i

$$\vec{p}^{\,tot} = \sum_{i=1}^{N} p_i, \quad \frac{d\vec{p}^{\,tot}}{dt} = F^{ext}_{tot}$$

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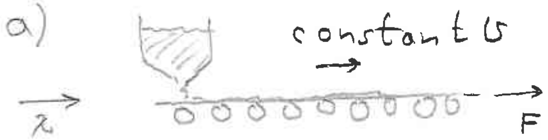
$\vec p^{\,tot} = \sum_{i=1}^{N} p_i$, $\quad \dfrac{d\vec p^{\,tot}}{dt} = F^{ext}_{tot}$

Thus, if $\vec F^{\,ext} = 0$ $\quad \dfrac{d\vec p^{\,tot}}{dt} = 0$

and the total momentum is conserved

Two examples

a)



Sand falls on the belt at the rate λ (gm/sec) what force F must be applied to the belt to keep it moving at a constant velocity v? Total horizontal momentum of all the sand

$$P_x^{tot}(t) = M_{on\ belt}(t)\, v + (M_{hopper} + M_{falling}) \times 0$$

$$\frac{dP_x^{tot}}{dt}(t) = \frac{d}{dt}\left[M_{on\ belt}(t)\right] v = \lambda v$$

$$= F \quad \Rightarrow \quad F = \lambda v$$

Important to identify a single collection of particles (all sand) and determine momentum of all of it!

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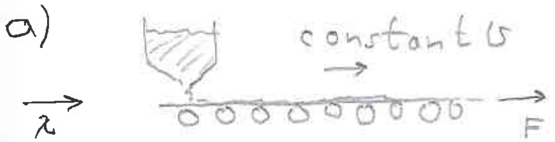

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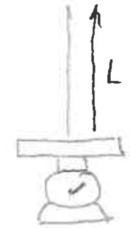
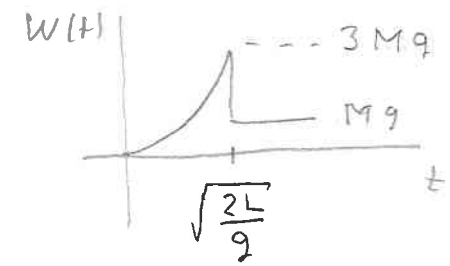


A chain of mass M and length L is hanging at rest just touching the platform of a scale. If the chain is released at $t=0$, what weight does the scale read as a function of time, $W(t)$?

$$P_z(t) = \frac{M}{L} \times \underbrace{\left(L - \tfrac{1}{2} g t^2\right)}_{\text{length of moving chain}} \underbrace{gt}_{\text{velocity of moving chain}}$$

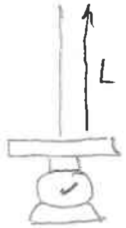
$$\frac{dP_z}{dt} = Mg - W(t)$$

$$Mg - \frac{3}{2}\frac{M}{L} g^2 t^2 = Mg - W(t)$$



$$W(t) = \begin{cases} \frac{3}{2}\frac{M}{L} g^2 t^2 & \frac{1}{2} g t^2 < L \\ Mg & \frac{1}{2} g t^2 > L \end{cases}$$

$$W\left(t = \sqrt{\frac{2L}{g}}\right) = 3Mg$$

b) 

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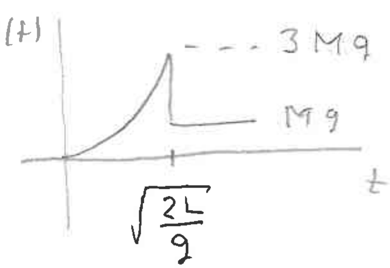

$$W\left(t=\sqrt{\frac{2L}{g}}\right) = 3Mg$$

2. Center of Mass: Useful concept closely related to total momentum

Introduce the massweighted average position:

$$\vec{R}_{cm} = \sum_{i=1}^{N}\frac{m_i}{M_{tot}}\vec{r}_i = \frac{\sum_{i=1}^{N} m_i\vec{r}_i}{\sum_{i=1}^{N} m_i}$$

is the position of center of mass for a collection of N particles where the i^{th} particle's mass is m_i and position is $\vec{r}_i$.

$$M_{tot}\underbrace{\frac{d\vec{R}_{cm}}{dt}}_{\vec{v}_{cm}} = M_{tot}\frac{\sum_{i=1}^{N} m_i\frac{d\vec{r}_i}{dt}}{M_{tot}} = \underbrace{\sum_{i=1}^{N} m_i\frac{d\vec{r}_i}{dt}}_{\vec{P}_{tot}}$$

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Center of mass moves like a point particle of which F^{ext}_{tot} has been applied!

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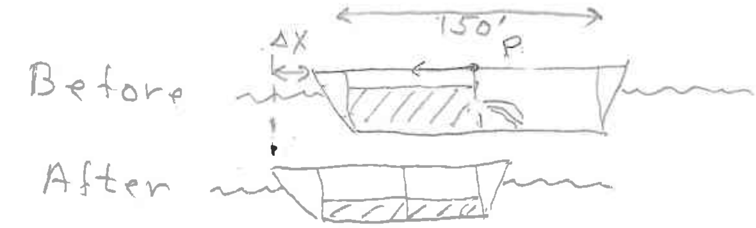
Of course our "point particles" are really composed of $\sim 10^{21}$ electrons and protons so earlier $\vec{r}_i$ we also center of mass positions themselves.

Example:



100 ton barge, 200 tons of oil. Oil flows from left compartment to fill both equally. How far does barge move?

Before $\quad x_{cm} = \dfrac{0 \times 100\text{ tons} - \frac{75'}{2}\, 200\text{ tons}}{100 + 300} + x_p^B$

After $\quad x_{cm}' = \dfrac{0 \cdot 100\text{ tons} + 0 \cdot 200\text{ tons}}{100 + 300} + x_p^A$

$$x_p^A - x_p^B = - \frac{\frac{75'}{2} \times 200}{300} = -25'$$

Since no horizontal external forces act on the barge, x_{cm} does not move

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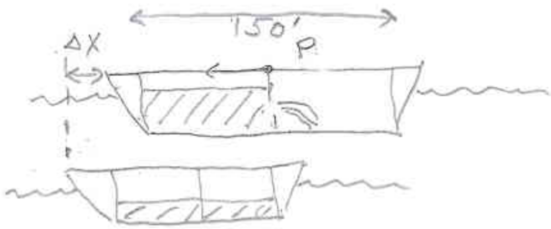

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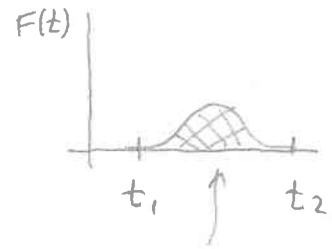
3. Impulse: since $\dfrac{d\vec{P}}{dt} = F(t)$

$$\int_{t_1}^{t_2} \frac{d\vec{P}}{dt}(t)\,dt = \vec{P}(t_2) - \vec{P}(t_1) = \Delta\vec{P}$$

$$= \int_{t_1}^{t_2} F(t)\,dt \equiv I \quad \Big\} \text{ impulse exerted by } F$$



during the interval between t_1 & t_2

area = impulse

D Energy

1. One-dimension

a) Assume $m\dfrac{d^2x}{dt^2} = F(x)$ where F depends only on x. Multiply both sides by $\dfrac{dx}{dt}$

$$m\frac{d^2x}{dt^2}\frac{dx}{dt} = F(x)\frac{dx}{dt}$$

Both terms can be written as time derivatives

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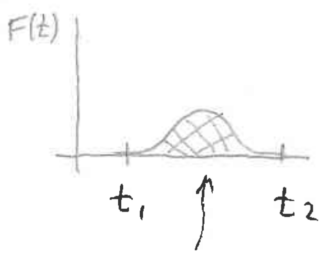

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Both terms can be written as time derivatives

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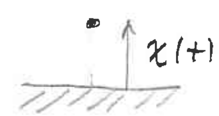
- $F(x)\frac{dx}{dt} = -\frac{d}{dt}U(x) = -\frac{dU}{dx}\frac{dx}{dt}$

if $\frac{dU}{dx}(x) = -F(x)$ or $U(x) = -\int^x F(x')\,dx'$

$$0 = \frac{dx}{dt}\left(m\frac{d^2x}{dt^2} - F(x)\right) = \frac{d}{dt}\left\{\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + U(x(t))\right\}$$

$$E = \underbrace{\frac{1}{2}m\left(\frac{dx}{dt}(t)\right)^2}_{\substack{\text{Kinetic}\\ \text{energy}}} + \underbrace{U(x(t))}_{\substack{\text{Potential}\\ \text{energy}}} \quad \text{is a constant}$$

b) Example: vertical motion near the surface of the earth



$$U(x) = -\int_0^x F(x')\,dx' = -\int_0^x (-mg)\,dx$$

$$= +mgx$$

$$E = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + mgx = \text{constant}$$

$$\frac{dx}{dt} = \pm\sqrt{2gx}$$

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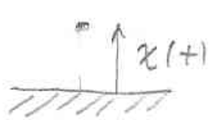

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If a particle is thown upward with initial velocity v_0, how high will it rise above its initial position

• $E_{botton} = \frac{1}{2}mv_0^2 + mgx_0$

$\parallel$

• $E_{top} = \cancelto{0}{\frac{1}{2}mv_{top}^2} + mg(x_0+h)$

$$h = v_0^2/2g$$

c) Use $E = \text{const}$ to solve for $x(t)$

$$E = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + U(x)$$

$$\frac{dx}{dt} = \sqrt{2(E-U(x))/m}$$

$$t = \int_0^t dt' = \int_{x(0)}^{x(t)} \frac{dt}{dx}\,dx = \int_{x(0)}^{x(t)} \frac{dx}{\sqrt{2(E-U(x))/m}}$$

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2. Can energy be used if F depends on more than position? Yes! Introduce "work" instead of potential. Again start with

$$\frac{d}{dt}\left\{ \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 \right\} = F(x(t), t) \frac{dx}{dt}$$

Integrate both sides wrt t

$$\int_{t_1}^{t_2} \frac{d}{dt}\left\{ \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 \right\} = \int_{t_1}^{t_2} F(x(t), t) \frac{dx}{dt}\, dt$$

$$\frac{1}{2} m v^2(t_2) - \frac{1}{2} m v^2(t_1) = \underbrace{\int_{x(t_1)}^{x(t_2)} F(x, t(x))\, dx}_{\text{work done by } F \text{ on } m}$$

$$= -\left[U(x_2) - U(x_1)\right],$$

if F depends only on x. Then work done by F is the potential enery lost.

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$$\frac{1}{2}m\,v^2(t_2) - \frac{1}{2}m\,v^2(t_1) = \underbrace{\int_{x(t_1)}^{x(t_2)} F(x,t(x))\,dx}_{\text{work done by F on m}}$$

$$= -\left[U(x_2) - U(x_1)\right],$$

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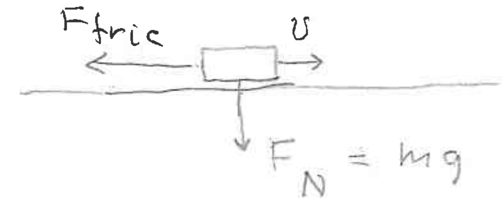
work done by F is the potential enery lost.

Often we can calculate the work done even when $U(x)$ does not exist. For example, sliding friction:



$$F_{fric} = \mu F_N \frac{v}{|v|}$$

Work done by friction as block slides from x_1 to x_2

$$W = \int_{x_1}^{x_2} F_{fric}\,dx = \int_{x_1}^{x_2} -\mu F_N \frac{v}{|v|}\,dx$$

$$= -(x_2 - x_1)\,\mu\, mg$$

Work is negative, K.E. is removed. If block starts with velocity v_0 how far does it slide

$$0 - \frac{1}{2}m v_0^2 = -(x_2 - x_1)\,\mu\, mg$$

$$x_2 - x_1 = v_0^2/\mu g$$