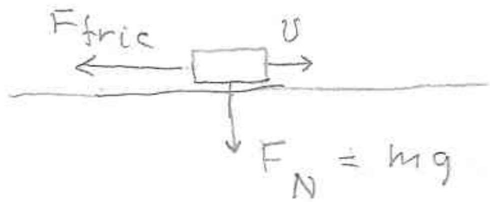
Often we can calculate the work done even when $\mathcal{U}(x)$ does not exist. For example, sliding friction:



$$F_{fric} = \mu F_N \frac{\upsilon}{|\upsilon|}$$

Work done by friction as block slides from x_1 to x_2

$$W = \int_{x_1}^{x_2} F_{fric}\, dx = \int_{x_1}^{x_2} -\mu F_N \frac{\upsilon}{|\upsilon|}\, dx$$

$$= -(x_2 - x_1)\mu m g$$

Work is negative, K.E. is removed. If block starts with velocity υ_0 how far does it slide

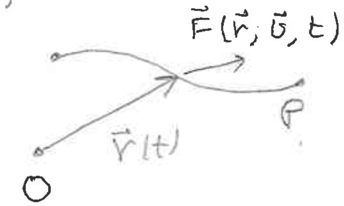
$$0 - \tfrac{1}{2} m \upsilon_0^2 = -(x_2 - x_1)\mu m g$$

$$x_2 - x_1 = \upsilon_0^2 / \mu g$$

3. Energy in 3 dimensions

a) Work and line integrals

$$m \frac{d^2 \vec{r}(t)}{dt^2} = F(\vec{r}(t), \vec{\upsilon}(t), t)$$



generalize 1-dim method and take the dot product with $\frac{d\vec{r}}{dt}$:

$$\frac{d^2\vec{r}}{dt^2} \cdot \frac{d\vec{r}}{dt} = \frac{d}{dt}\, \frac{1}{2}\left(\frac{d\vec{r}}{dt} \cdot \frac{d\vec{r}}{dt}\right) = \frac{d}{dt}\left[\frac{1}{2}\vec{\upsilon}^2\right]$$

just as for 1-dim. Thus

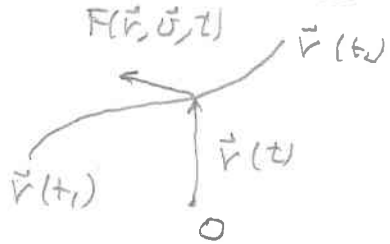
$$\frac{d}{dt}\left(\frac{1}{2} m \vec{\upsilon}^2\right) = \vec{F}(\vec{r}(t), \vec{\upsilon}(t), t) \cdot \frac{d\vec{r}}{dt}$$

Integrate both sides from t_1 to t_2

$$\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{1}{2} m \vec{\upsilon}^2(t)\right) = \int_{t_1}^{t_2} \vec{F}(\vec{r}(t), \vec{\upsilon}(t), t)\, \frac{d\vec{r}}{dt}\, dt$$

$$\underbrace{\frac{1}{2} m \vec{\upsilon}^2(t_2) - \frac{1}{2} m \vec{\upsilon}^2(t_1)}_{\text{change in K.E.}} = \underbrace{\int_P \vec{F}(\vec{r}, \vec{\upsilon}(\vec{r}), t(\vec{r})) \cdot d\vec{r}}_{\text{Work done = Line integral}}$$

$\int_P \vec{F}(\vec{r}, \vec{v}(\vec{r}), t(\vec{r})) \cdot d\vec{r}$ = line integral of $\vec{F}$ along the path P

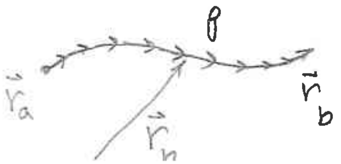


- Recall Riemann sum definition of integral

$$\int_{x_a}^{x_b} f(x)\,dx = \lim_{\substack{\Delta x_n \to 0 \\ N \to \infty}} \sum_{n=1}^{N} f(x_n)\,\Delta x_n$$

$$x_1 = x_a, \quad x_{N+1} = x_b, \quad \Delta x_n = x_{n+1} - x_n$$

- Do the same thing to define the line integral



Choose $N+1$ points $\vec{r}_i$ $\quad 1 \le i \le N+1$ on the path P

$$\vec{r}_1 = \vec{r}_a, \quad \vec{r}_{N+1} = \vec{r}_b \qquad \Delta\vec{r}_n = \vec{r}_{n+1} - \vec{r}_n$$

$$\int_P \vec{F}(\vec{r}) \cdot d\vec{r} \equiv \lim_{\substack{\Delta \vec{r}_n \to 0 \\ N \to \infty}} \sum_{n=1}^{N} \vec{F}(\vec{r}_n) \cdot \Delta\vec{r}_n$$

A wonderful abstract concept!

Need to express as ordinary integrals

Two methods:

i) Introduce a parameter α:

$\alpha_a \le \alpha \le \alpha_\beta$ and a function $\vec{r}(\alpha)$ which traces out the path P from $\vec{r}(\alpha_a) = \vec{r}_a$ to $\vec{r}(\alpha_b) = \vec{r}_b$

$$\int_P \vec{F}(\vec{r}) \cdot d\vec{r} = \int_{\alpha_a}^{\alpha_b} \vec{F}(r(\alpha)) \cdot \frac{d\vec{r}(\alpha)}{d\alpha}\, d\alpha$$

a standard, definite integral

ii) Use components

$$\int_P \vec{F}(x) \cdot d\vec{r} = \int_P \left[F_x\,dx + F_y\,dy + F_z\,dz \right]$$

$$= \int_{x_a}^{x_b} F_x(x, y(x), z(x))\,dx$$

$$+ \int_{y_a}^{y_b} F_y(x(y), y, z(y))\,dy$$

$$+ \int_{z_a}^{z_b} F_z(x(z), y(z), z)\,dz$$

$\uparrow \quad \uparrow$

may need too evaluate path in segments to make coordinates single-valued

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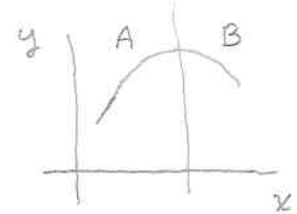
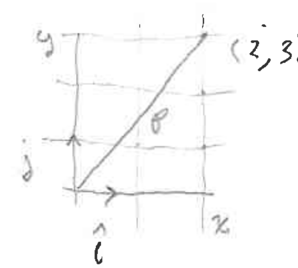


$y(x)$ OK, $x(y)$ defined differently in A & B

b) Example: $\vec{F}(x,y) = a y^2 \hat{i} + b x \hat{j}$

(use $\hat{i}, \hat{j}, \hat{k}$ as our orthonormal basis)

How much work is done by $\vec{F}$ acting on the mass as it is moved from $(0,0)$ to $(2,3)$



$y(x) = \frac{3}{2}x$

$x(y) = \frac{2}{3}y$

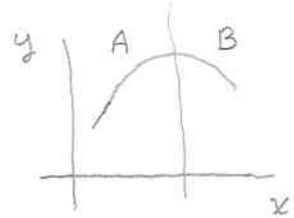
$$W = \int_P \vec{F} \cdot dr = \int_P F_x\, dx + F_y\, dy$$

$$= \int_0^2 a y^2(x)\, dx + \int_0^3 b\, x(y)\, dy$$

$$= \int_0^2 a \left(\frac{3}{2}x\right)^2 dx + \int_0^3 b \left(\frac{2}{3}y\right) dy$$

$$= a \frac{9}{4} \frac{2^3}{3} + b \frac{2}{3} \frac{3^2}{2}$$

$$= 3a + 3b$$



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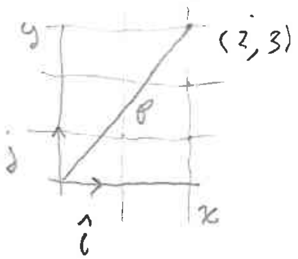

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$y(x) = \frac{3}{2}x$

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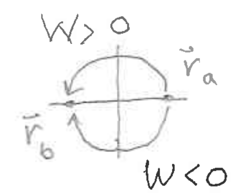
c) If $\vec{F}(\vec{r})$ depends only on $\vec{r}$ can we find a potential energy $U(\vec{r})$ so that

$$\int_{\vec{r}_a}^{\vec{r}_b} \vec{F}(\vec{r}) \cdot d\vec{r} = -\left[U(\vec{r}_a) - U(\vec{r}_a)\right]$$

as we could in one dimension?

NO!

Look at $\vec{F}(\vec{r}) = \hat{\theta} f(r)$ $\quad f(r) > 0$



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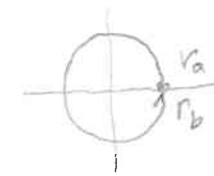

$$\oint \vec{F}(r) \cdot d\vec{r} = 2\pi r\, f(r) > 0$$

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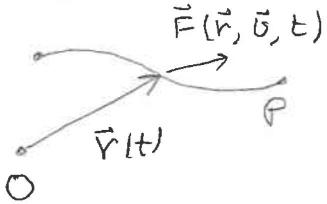

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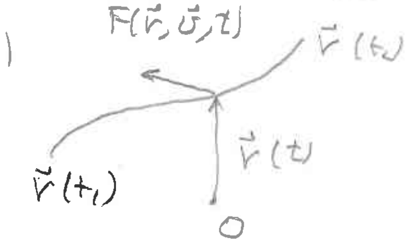

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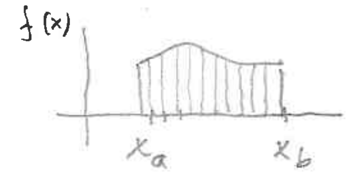

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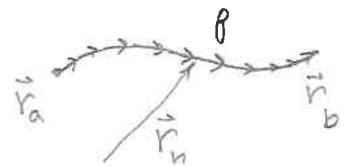

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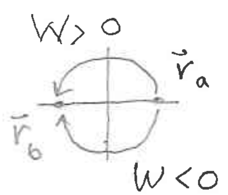

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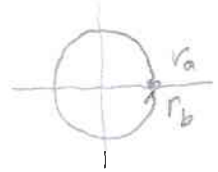

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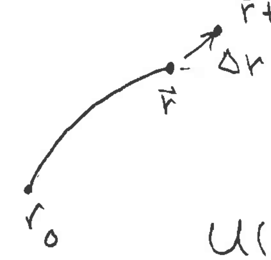
Define $U(\vec{r}, \vec{r}_0) = -\int_{\vec{r}_0}^{r} \vec{F}(\vec{r}')\cdot d\vec{r}'$ ← does not depend on path



$$U(\vec{r}+\Delta\vec{r}, r_0) - U(\vec{r}, r_0) = -\int_{\vec{r}}^{\vec{r}+\Delta\vec{r}} F(\vec{r})\cdot d\vec{r}$$

<u>for small $\Delta\vec{r}$</u> $\quad \approx -\left[F_x(\vec{r})\,\Delta x + F_y(r)\,\Delta y + F_z(r)\,\Delta z\right]$

Look at $U(x+\Delta x, y+\Delta y, z+\Delta z) - U(x,y,z)$

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$$\approx \frac{dU}{dx}(x,y,z)\,\Delta x + \frac{dU}{dy}(x,y,z)\,\Delta y + \frac{dU}{dz}(x,y,z)\,\Delta z$$

$\frac{\partial U}{\partial x} \equiv$ "partial" derivative, hold y & z fixed

Since Δx, Δy and Δz are independent we can equate their coefficients:

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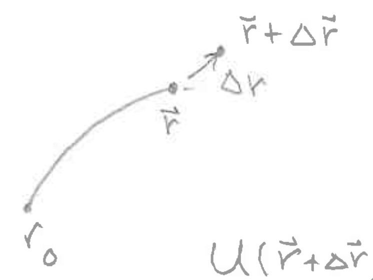

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$$\frac{\partial U}{\partial x} = -F_x, \quad \frac{\partial U}{\partial y} = -F_y, \quad \frac{\partial U}{\partial z} = -F_z$$

Two important consequences:

i) Since $F_x \hat e_x + F_y \hat e_y + F_z \hat e_z$ defines a vector so must

$$\vec\nabla U \equiv \frac{\partial U}{\partial x}\hat e_x + \frac{\partial U}{\partial y}\hat e_y + \frac{\partial U}{\partial z}\hat e_z$$

since $\vec\nabla U = -\vec F$. Given a function $U(\vec r)$ we can define a vector $\vec\nabla U$, the "gradient" of U

ii $\vec F = -\vec\nabla U$ determines the three function $F_x(\vec r)$, $F_y(\vec r)$ & $F_z(\vec r)$ from one function $U(\vec r)$!

Thus path independence of the work $\int_P^{\vec r} \vec F(\vec r\,') \cdot d\vec r\,'$ is a very strong condition!

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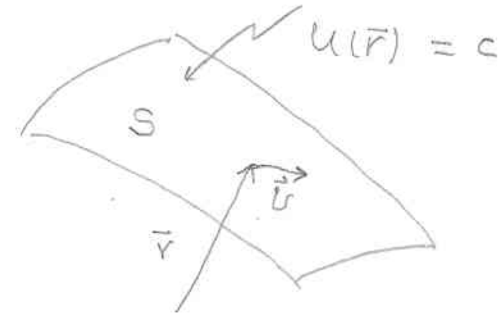
Meaning of $\vec{\nabla} U$:



Consider a surface S of constant $U(\vec{r}) = c$

If $\vec{v}$ is tangent to S at $\vec{r}$, the $U(\vec{r} + \Delta r\, \vec{v}) - U(\vec{r}) \sim (\Delta r)^2$ for small Δr

But $U(\vec{r} + \Delta r \vec{v}) - U(\vec{r})$

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The gradient of U is perpendicular to surfaces of constant U.

Since $U(\vec{r} + \Delta r \vec{v}) - U(\vec{r}) = \vec{\nabla} U \cdot \vec{v}\, \Delta r$, choosing $\vec{v} \parallel \vec{\nabla} U$ will give the largest ΔU: $\vec{\nabla} U$ points in the direction of the most rapid change of U.

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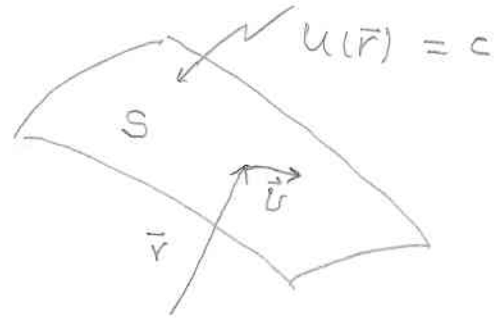

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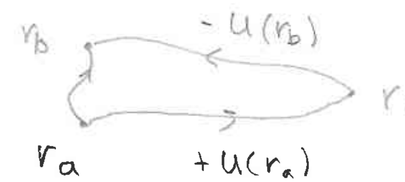
A force for which the work done does not depend on path is a conservative force. We can define

$$U(\vec{r}) = -\int_{r_0}^{\vec{r}} \vec{F}(\vec{r}) \cdot d\vec{r}$$

Then

$$\int_{r_a}^{r_b} \vec{F}(\vec{r}) \cdot d\vec{r} = -\left[U(r_b) - U(r_a)\right]$$

since



and $\vec{F}(\vec{r}) = -\vec{\nabla} U(r).$

Use paramethod to evaluate

$$\int_{\alpha_a}^{\alpha_b} \vec{F}(\vec{r}(\alpha)) \cdot \frac{d\vec{r}}{d\alpha}\, d\alpha = -\int_{\alpha_a}^{\alpha_b} \left[\frac{\partial U}{\partial x}\frac{dx}{d\alpha} + \frac{\partial U}{\partial y}\frac{dy}{d\alpha} + \frac{\partial U}{\partial z}\frac{dz}{dx}\right] d\alpha$$

$$= -\int_{\alpha_a}^{\alpha_b} \frac{d}{d\alpha} U(x(\alpha), y(\alpha), z(\alpha))\, d\alpha$$

$$= -\left[U(r_b) - U(r_a)\right] \quad \checkmark$$

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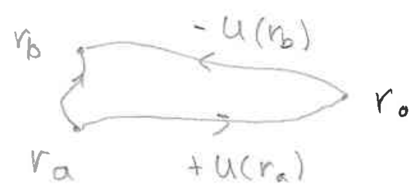
Then $\int_{r_a}^{r_b} \vec{F}(\vec{r})\cdot d\vec{r} = -\left[U(r_b) - U(r_a)\right]$

since



and $\vec{F}(\vec{r}) = -\vec{\nabla} U(r)$.

Use parame thod to evaluate

$$\int_{\alpha_a}^{\alpha_b} \vec{F}(\vec{r}(\alpha))\cdot\frac{d\vec{r}}{d\alpha}\, d\alpha = -\int_{\alpha_a}^{\alpha_b}\left[\frac{\partial U}{\partial x}\frac{dx}{d\alpha} + \frac{\partial U}{\partial y}\frac{dy}{d\alpha} + \frac{\partial U}{\partial z}\frac{dz}{dx}\right]d\alpha$$

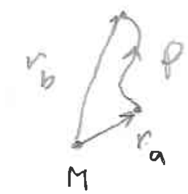
$$= -\int_{\alpha_a}^{\alpha_b} \frac{d}{d\alpha} U(x(\alpha), y(\alpha), z(\alpha))\, d\alpha$$

$$= -\left[U(r_b) - U(r_a)\right] \checkmark$$

Important example: gravity

For simplicity locate origin at M. Force on m:

$$F(\vec{r}) = -\hat{r}\,\frac{GMm}{r^2}$$



Does $\int_{\vec{r}_a}^{\vec{r}_b} \vec{F}(\vec{r})\cdot d\vec{r}$ depend on P?

$$\int_{\vec{r}_a}^{\vec{r}_b} -\frac{GMm}{r^2}\underbrace{\hat{r}\cdot d\vec{r}}_{dr} = \int_{r_a}^{r_b} -\frac{GMm}{r^2}\,dr =$$

$$= +\frac{GMm}{r_b} - \frac{GMm}{r_a}$$

Use $U(\vec{r}) = -\frac{GMm}{|r|}$ choosing constant so that $\lim_{|\vec{r}|\to\infty} U(\vec{r}) = 0$