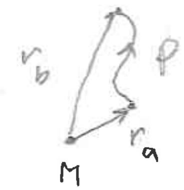
10/1/20 (58)

e) Important example: gravity

For simplicity locate origin at M. Force on m:

$$F(\vec{r}) = -\hat{r}\,\frac{GMm}{r^2}$$



Does $\int_{\vec{r}_a}^{\vec{r}_b} \vec{F}(\vec{r})\cdot d\vec{r}$ depend on P?

$$\int_{\vec{r}_a}^{\vec{r}_b} -\frac{GMm}{r^2}\,\underbrace{\hat{r}\cdot d\vec{r}}_{dr} = \int_{r_a}^{r_b} -\frac{GMm}{r^2}\,dr =$$

$$= +\frac{GMm}{r_b} - \frac{GMm}{r_a}$$

Use $U(\vec{r}) = -\frac{GMm}{|\vec{r}|}$ choosing constant so that $\lim_{|\vec{r}|\to\infty} U(\vec{r}) = 0$

Does this work?

$$-\vec{\nabla}\left[-\frac{GMm}{|\vec{r}|}\right] \overset{?}{=} \vec{F} = -\frac{GMm}{r^2}\hat{r}$$

(59)

Calculate F_x:

$$-\frac{\partial}{\partial x}\left[-\frac{GMm}{|\vec{r}|}\right] = +\frac{\partial}{\partial x}\frac{GMm}{\sqrt{x^2+y^2+z^2}}$$

$$= -\frac{1}{2}\frac{GMm\ 2x}{[x^2+y^2+z^2]^{3/2}}$$

$$= -\frac{GMm}{r^2}\frac{x}{r} = -\frac{GMm}{r^2}\hat{r}_x \quad ✓$$

Problem: What is the smallest vertical velocity that must be given to an object if it is to escape from Earth?

need $E = \frac{1}{2}m\vec{v}^2 - \frac{GMm}{r_E} \geq 0$

$$v \geq \left[\frac{2GM}{r_E}\right]^{1/2} = \left[2\underbrace{\frac{GM}{r_E^2}}_{g}\cdot r_E\right]^{1/2}$$

$$= \left[2\cdot 32\ \frac{ft}{sec^2}\times 3,960\ mi\right]^{1/2}$$

$$= \left[2\cdot 32\ \frac{1}{5,280}\ 3,960\right]\frac{mi}{sec}\times\frac{3600\ sec}{hr}$$

$$= 24,900\ mi/hr$$

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$$= -\frac{GMm}{r^2}\frac{x}{r} = -\frac{GMm}{r^2}\hat r_x \quad \checkmark$$

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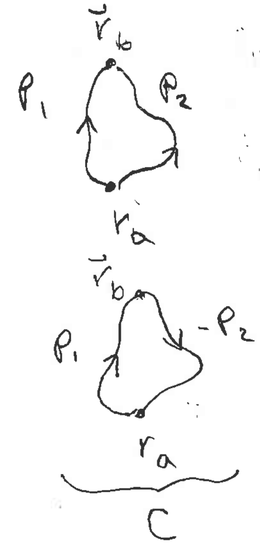
f) What condition must $\vec F(\vec r)$ obey for $\int_{\vec r_a}^{\vec r_b}\vec F(\vec r)\cdot d\vec r$ to be path independent

or for $\vec F(\vec r) = -\vec\nabla U(\vec r)$ for some $U(\vec r)$?

Consider two paths P_1 & P_2

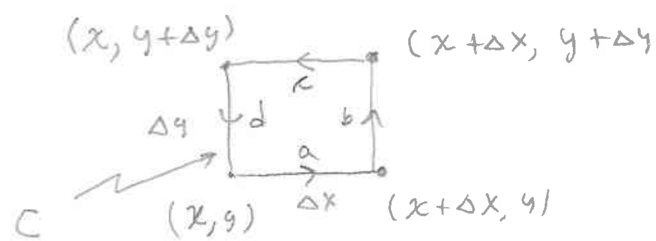


$$\int_{\vec r_a, P_1}^{\vec r_b}\vec F(\vec r)\cdot d\vec r = \int_{r_a, P_2}^{\vec r_b}\vec F(\vec r)\cdot d\vec r$$

$$\Longleftrightarrow$$

$$\oint_C \vec F(\vec r)\cdot d\vec r = 0$$

A clever idea: look at very small loops. Start in 2-dim



$$0 = \oint_C \vec F(\vec r)\cdot d\vec r$$

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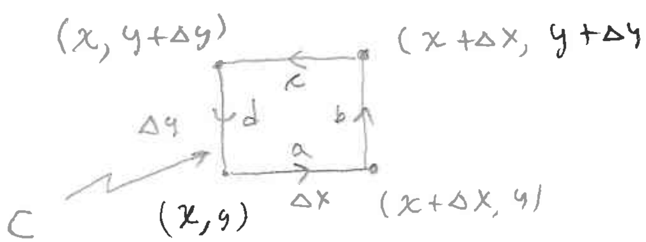

$$0 = \oint_C \vec{F}(\vec{r})\cdot d\vec{r}$$

Use derivatives to evaluate $\lim \Delta x, \Delta y \to 0$:

$$\oint_C \vec{F}(\vec{r})\cdot d\vec{r} = \overset{(a)}{\int_x^{x+\Delta x} F_x(x', y)\, dx'} + \overset{(b)}{\int_y^{y+\Delta y} F_y(x+\Delta x, y')\, dy'}$$

$$- \overset{(c)}{\int_x^{x+\Delta x} F_x(x', y+\Delta y)\, dx'} - \overset{(d)}{\int_y^{y+\Delta y} F_y(x, y')\, dy}$$

$$= -\int_x^{x+\Delta x} \left[F_x(x', y+\Delta y) - F_x(x', y)\right] dx' + \int_y^{y+\Delta y} \left[F_y(x+\Delta x, y') - F_y(x, y')\right] dx'$$

$$\approx -\Delta y \int_x^{x+\Delta x} \frac{\partial F_x}{\partial y}(x', y)\, dx' + \Delta x \int_y^{y+\Delta y} \frac{\partial F_y}{\partial x}(x, y')\, dy'$$

$$\approx \left[\frac{\partial F_y}{\partial x}(x,y) - \frac{\partial F_x}{\partial y}(x,y)\right] \Delta x \Delta y = 0$$

Thus

$$\oint_C \vec{F}(\vec{r})\cdot d\vec{r} = 0 \Rightarrow \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0$$

$$\overset{?}{\Longleftarrow}$$

Can this logic be reversed? Yes!

Use derivatives to evaluate

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$$- \overset{\text{(c)}}{\int_x^{x+\Delta x} F_x(x', y+\Delta y)\,dx'} - \overset{\text{(d)}}{\int_y^{y+\Delta y} F_y(x,y')\,dy}$$

$$= - \int_x^{x+\Delta x} \left[F_x(x', y+\Delta y) - F_x(x',y)\right] dx' + \int_y^{y+\Delta y} \left[F_y(x+\Delta x, y') - F_y(x,y')\right] dx'$$

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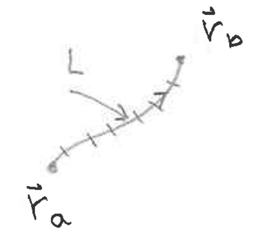
Prove using Stokes' theorem:

1st Lines



the points B and A make up the boundary of L

"boundary of" $\partial L = \{\vec r_b, \vec r_a\}$

tail

head

Break L into n segments l_i $1 \le i \le N$

$$\partial l_i = \{\vec r_{i+1}, \vec r_i\} \qquad L = \bigcup_{i=1}^{N} l_i$$

We can evaluate $U(\vec r)$ on the boundary of L:

$$U\Big|_{\partial L} \equiv U(r_b) - U(r_a) = \sum_{i=1}^{N} U\Big|_{\partial l_i}$$

$$= \sum_{i=1}^{N} \left[U(\vec r_{i+1}) - U(\vec r_i)\right]$$

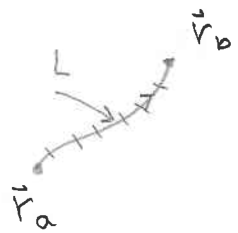
$$\approx \sum_{i=1}^{N} \vec\nabla U(\vec r_i)\cdot \underbrace{[\vec r_{i+1} - \vec r_i]}_{\Delta \vec r_i}$$

$$= \int_L \vec\nabla U(r)\cdot d\vec r$$

Stokes' theorem for 1-dim lines

Prove using Stokes' theorem:

1^{st} Lines



the points B and A make up the boundary of L

"boundary of" $\partial L = \{\vec{r}_b, \vec{r}_a\}$ ← tail, head

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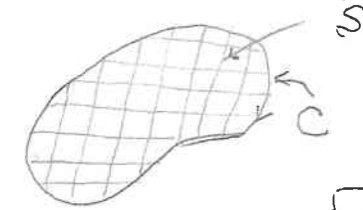
Stokes' theorem for 1-dim lines

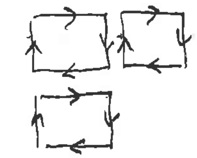
We need Stokes' theorem for 2 dim.

$\oint_C \vec{F}(\vec{r}) \cdot d\vec{r} = 0$ view the closed curve C as the boundary of a surface S



Now

$$F\Big|_{\partial S} = \oint_C \vec{F}(\vec{r}) \cdot d\vec{r}$$



Divide S into small squares

$$S = \bigcup_i S_i$$

$$\oint_{\partial S} \vec{F}(\vec{r}) \cdot d\vec{r} = \sum_i \underbrace{\oint_{\partial S_i} \vec{F}(\vec{r}) \cdot d\vec{r}}_{\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \Delta S_i}$$

internal boundaries cancel

area of

$$= \int \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dS$$

Line integral = surface integral

$$\oint_C \vec{F}(\vec{r}) \cdot d\vec{r} = 0 \Leftarrow \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0$$

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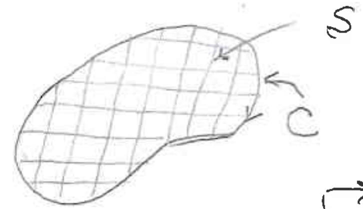

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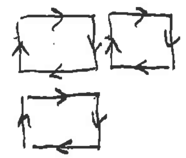

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We have two tasks:

i) Define surface integrals

ii) Generalize to curved 2-dim surfaces in 3 dim.

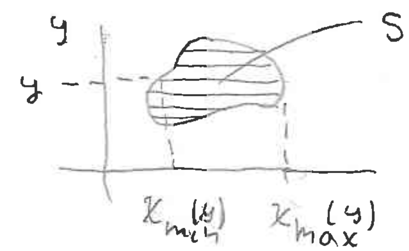
Surface integrals:

Easy concept; given a surface S divide it into same bit S_i $1 \le i \le N$ each with area ΔS_i and define

$$\int_S f(x,y)\, dS \equiv \lim_{\substack{\Delta S_i \to 0 \\ N\to\infty}} f(\underbrace{x_i, y_i}_{})\,\Delta S_i$$

Locates a point on S_i

Best to view as a multiple integral



Divide into strips,

Integrate over each strip.

Integrate the strips

$$\int_S f(x,y)\, dS = \int_{y_{min}}^{y_{max}} dy \left\{ \int_{x_{min}(y)}^{x_{max}(y)} dx\, f(x,y) \right\}$$

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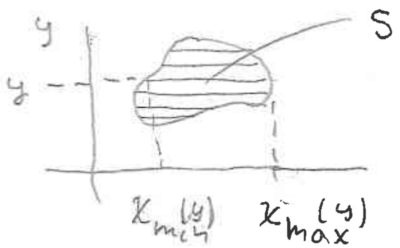

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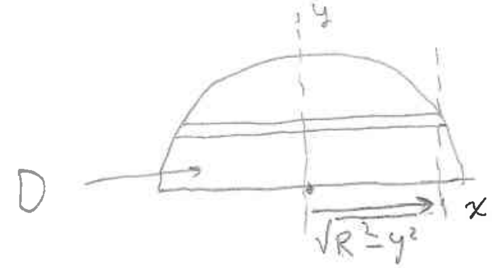
Example:

Find the location of the center of mass of a half-disk of radius R



$x_{cm} = 0$, $y_{cm} = ?$

$$y_{cm} = \frac{1}{M} \int_D y\, ds \times \underbrace{\frac{M}{\pi R^2/2}}_{\text{density of mass per unit area}}$$

$$= \frac{2}{\pi R^2} \int_0^R dy\, y \int_{-\sqrt{R^2-y^2}}^{+\sqrt{R^2-y^2}} dx \quad , \; ds = dx\,dy$$

$$= \frac{2}{\pi R^2} \int_0^R dy\, y \cdot 2\sqrt{R^2-y^2}$$

$$= -\frac{4}{\pi R^2} \frac{1}{2} \frac{2}{3} \left[R^2 - y^2\right]^{3/2} \Big|_0^R = +\frac{4}{3} \frac{1}{\pi R^2} R^3$$

$$= \frac{4}{3\pi} R$$

$$\vec{r}_{cm} = (x_{cm}, y_{cm}) = \left(0, \frac{4}{3\pi} R\right)$$

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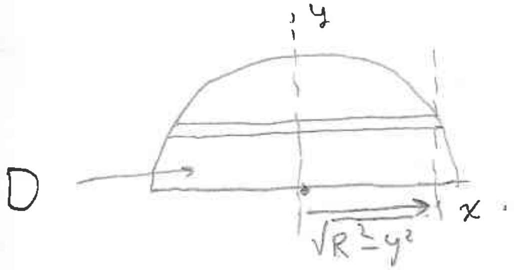

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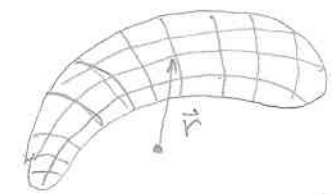
Stokes' theorem for 2-D surfaces in 3-dim



For a smooth surface choose at a point $\vec{r}$ use a plane tangent at $\vec{r}$ which we could choose as x-y plane and use earlier derivation:

$$\oint_{\partial(\Delta S)} \vec{F}(\vec{r})\cdot d\vec{r} = \Delta S\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)(\vec{r})$$

which we must transform into a single common coordinate system.

Recognize that $\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$ is the z component of a vector, the curl of $\vec{F}$

$$(\vec{\nabla}\times\vec{F})_x = \frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y$$

$$(\vec{\nabla}\times\vec{F})_y = \frac{\partial}{\partial z}F_x - \frac{\partial}{\partial x}F_z$$

$$(\vec{\nabla}\times\vec{F})_z = \frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x$$

$x \to y \to z$ cycle through this order

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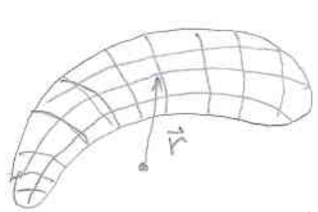

For a smooth surface choose at a point $\vec r$ use a plane tangent at $\vec r$ which we could choose as x-y plane and use earlier derivation:

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$$(\vec\nabla \times \vec F)_x = \frac{\partial}{\partial y} F_z - \frac{\partial}{\partial z} F_y$$

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$$(\vec\nabla \times \vec F)_z = \frac{\partial}{\partial x} F_y - \frac{\partial}{\partial y} F_x$$

$x \to y \to z$

cycle through this order

Now for our tangent plane at $\vec r$ chosen to be the x-y plane we can express

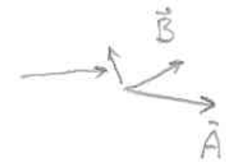
$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = (\vec\nabla \times \vec F)_z = (\vec\nabla \times \vec F)\cdot \underbrace{\hat n}_{\text{unit vector normal to } S}$$

$$\oint_C \vec F \cdot d\vec r = \int dS\, (\vec\nabla \times \vec F)\cdot \hat n$$

direction given by right-hand rule.

To see that $\vec\nabla \times \vec F$ is really a vector we must define the cross product of two vectors $\vec A$, $\vec B$:

$\vec A \times \vec B$



$\vec A \times \vec B$ is perpendicular to $\vec A$ & $\vec B$, its direction is given by the right hand rule and its length is given by

$$|\vec A \times \vec B| = |\vec A|\,|\vec B|\,|\sin\theta|$$

clearly defines a vector.

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$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = (\vec{\nabla}\times\vec{F})_z = (\vec{\nabla}\times\vec{F})\cdot\underbrace{\hat{n}}_{\substack{\text{unit vector}\\ \text{normal to } S}}$$

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$$|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}||\sin\theta|$$

clearly defines a vector.

Claim the coordinates of $\vec{A}\times\vec{B}$ are

$$(\vec{A}\times\vec{B})_x = A_y B_z - A_z B_y$$
$$(\vec{A}\times\vec{B})_y = A_z B_x - A_x B_z$$
$$(\vec{A}\times\vec{B})_z = A_x B_y - A_y B_x$$

note

$$\vec{C}\cdot(\vec{A}\times\vec{B}) = \det\begin{pmatrix} C_x & C_y & C_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{pmatrix}$$

just the dot product

$$= C_x(A_y B_z - A_z B_y) + C_y(A_z B_x - A_x B_z) + C_z(A_x B_y - A_y B_x)$$

$\Rightarrow \vec{A}\times\vec{B} \perp \vec{A}$ & $\vec{B}$

Need "Back-Cab" formula

$$\vec{A}\times(\vec{B}\times\vec{C}) = \underbrace{\vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})}_{\text{perpendicular to } \vec{A} \text{ \& } \vec{B}\times\vec{C}\ \checkmark}$$

check x component

$$A_y(B_x C_y - B_y C_x) - A_z(B_z C_x - B_x C_z)$$

$$\stackrel{?}{=} B_x(A_x C_x + A_y C_y + A_z C_z) - C_x(A_x B_x + A_y B_y + A_z B_z)$$

Claim the coordinates of $\vec{A}\times\vec{B}$ are

$$(\vec{A}\times\vec{B})_x = A_y B_z - A_z B_y$$

$$(\vec{A}\times\vec{B})_y = A_z B_x - A_x B_z$$

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$$= C_x(A_y B_z - A_z B_y) + C_y(A_z B_x - A_x B_z) + C_z(A_x B_y - A_y B_x)$$

$\Rightarrow \vec{A}\times\vec{B} \perp \vec{A}$ & $\vec{B}$

Need "Back-Cab" formula

$$A\times(\vec{B}\times\vec{C}) = \underbrace{\vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(A\cdot B)}_{\text{perpendicular to } \vec{A} \text{ \& } \vec{B}\times\vec{C}} \checkmark$$

check x component

$$A_y(B_x C_y - B_y C_x) - A_z(B_z C_x - B_x C_z)$$

$$\stackrel{?}{=} B_x(A_x C_x + A_y C_y + A_z C_z) - C_x(A_x B_x + A_y B_y + A_z C_z)$$

Finally check

$$|\vec{A}\times\vec{B}|^2 = (\vec{A}\times\vec{B})\cdot(\vec{A}\times\vec{B})$$

$$= \vec{A}\cdot(\vec{B}\times(\vec{A}\times\vec{B}))$$

$$= \vec{A}\cdot(\vec{A}(\vec{B}\cdot\vec{B}) - \vec{B}(A\cdot\vec{B}))$$

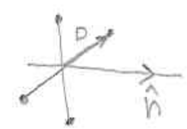
$$= |A|^2|B|^2 - |(A\cdot B)|^2 \checkmark$$

$$= |A|^2|B|^2(1-\cos^2\theta) = |A|^2|B|^2\sin^2\theta$$

Thus, if $\vec{F}(\vec{r})$ is a vector so is $\vec{\nabla}\times\vec{F}$.

Can we understand curl?

Use Purcell's curl meter



Let $\vec{F}(\vec{r})$ act on all four masses

$D^2\,\hat{n}\cdot(\vec{\nabla}\times\vec{F})$ = torque $\vec{F}$ exerts about $\hat{n}$.

Return later after studying torque