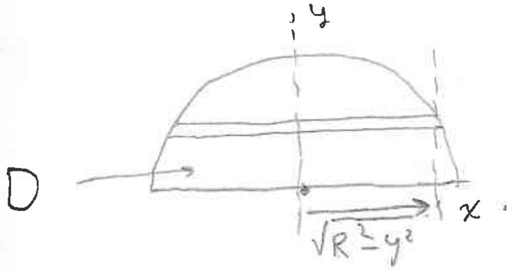
Example:

Find the location of the center of mass of a half-disk of radius R



$x_{cm} = 0$, $y_{cm} = ?$

$$y_{cm} = \frac{1}{M}\int_D y \, ds \times \underbrace{\frac{M}{\pi R^2/2}}_{\substack{\text{density of}\\ \text{mass per}\\ \text{unit area}}}$$

$$= \frac{2}{\pi R^2}\int_0^R dy \, y \int_{-\sqrt{R^2-y^2}}^{+\sqrt{R^2-y^2}} dx \quad , \; ds = dx\,dy$$

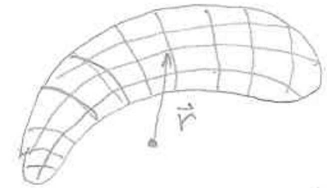
$$= \frac{2}{\pi R^2}\int_0^R dy \cdot 2\sqrt{R^2-y^2}$$

$$= -\frac{4}{\pi R^2}\frac{1}{3}\left[R^2-y^2\right]^{3/2}\Big|_0^R = +\frac{4}{3}\frac{1}{\pi R^2}R^3$$

$$= \frac{4}{3\pi}R$$

$$\vec{r}_{cm} = (x_{cm}, y_{cm}) = \left(0, \frac{4}{3\pi}R\right)$$

Stokes' theorem for 2-D surfaces in 3-dim



For a smooth surface choose at a point $\vec{r}$ use a plane tangent at $\vec{r}$ which we could choose as x-y plane and use earlier derivation:

$$\oint_{\partial(\Delta S)} \vec{F}(\vec{r})\cdot d\vec{r} = \Delta S\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)(\vec{r})$$

which we must transform into a single common coordinate system.

Recognize that $\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$ is the z component of a vector, the curl of $\vec{F}$

$$(\vec{\nabla}\times\vec{F})_x = \frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y$$

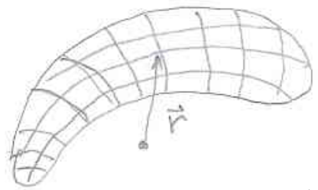
$$(\vec{\nabla}\times\vec{F})_y = \frac{\partial}{\partial z}F_x - \frac{\partial}{\partial x}F_z$$

$$(\vec{\nabla}\times\vec{F})_z = \frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x$$

$x \to y \to z$

cycle through this order

Stokes' theorem for 2-D surfaces in 3-dim



For a smooth surface choose at a point $\vec{r}$ use a plane tangent at $\vec{r}$ which we could choose as x-y plane and use earlier derivation:

$$\oint_{\partial(\Delta S)} \vec{F}(\vec{r}) \cdot d\vec{r} = \Delta S \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)(\vec{r})$$

which we must transform into a single common coordinate system.

Recognize that $\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$ is the z component of a vector, the curl of $\vec{F}$

$$(\vec{\nabla} \times \vec{F})_x = \frac{\partial}{\partial y} F_z - \frac{\partial}{\partial z} F_y$$

$$(\vec{\nabla} \times \vec{F})_y = \frac{\partial}{\partial z} F_x - \frac{\partial}{\partial x} F_z$$

$$(\vec{\nabla} \times \vec{F})_z = \frac{\partial}{\partial x} F_y - \frac{\partial}{\partial y} F_x$$

$x \to y \to z$

cycle through this order

Now for our tangent plane at $\vec{r}$ chosen to be the x-y plane we can express

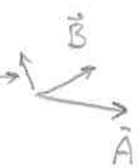
$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = (\vec{\nabla} \times \vec{F})_z = (\vec{\nabla} \times \vec{F}) \cdot \underbrace{\hat{n}}_{\text{unit vector normal to } S}$$

$$\oint_C \vec{F} \cdot d\vec{r} = \int ds \, (\vec{\nabla} \times \vec{F}) \cdot \hat{n}$$

direction given by right-hand rule.

To see that $\vec{\nabla} \times \vec{F}$ is really a vector we must define the cross product of two vectors $\vec{A}$, $\vec{B}$:

$\vec{A} \times \vec{B}$



$\vec{A} \times \vec{B}$ is perpendicular to $\vec{A}$ & $\vec{B}$, its direction is given by the right hand rule and its length is given by

$$|\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}| |\sin \theta|$$

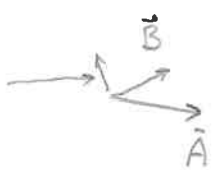
clearly defines a vector.

Now for our tangent plane at $\vec{r}$ chosen to be the x-y plane we can express

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = (\vec{\nabla}\times\vec{F})_z = (\vec{\nabla}\times\vec{F})\cdot\underbrace{\hat{n}}_{\text{unit vector normal to } S}$$

& $$\oint_C \vec{F}\cdot d\vec{r} = \int ds\,(\vec{\nabla}\times\vec{F})\cdot\hat{n}$$ direction given by right-hand rule.

To see that $\vec{\nabla}\times\vec{F}$ is really a vector we must define the cross product of two vectors $\vec{A}$, $\vec{B}$:

$\vec{A}\times\vec{B}$ 

$\vec{A}\times\vec{B}$ is perpendicular to $\vec{A}$ & $\vec{B}$, its direction is given by the right hand rule and its length is given by

$$|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}||\sin\theta|$$

clearly defines a vector.

Claim the coordinates of $\vec{A}\times\vec{B}$ are

$$(\vec{A}\times\vec{B})_x = A_y B_z - A_z B_y$$

$$(\vec{A}\times\vec{B})_y = A_z B_x - A_x B_z$$

$$(\vec{A}\times\vec{B})_z = A_x B_y - A_y B_x$$

note $$\vec{C}\cdot(\vec{A}\times\vec{B}) = \det\begin{pmatrix} C_x & C_y & C_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{pmatrix}$$

just the dot product
$$= C_x(A_y B_z - A_z B_y) + C_y(A_z B_x - A_x B_z) + C_z(A_x B_y - A_y B_x)$$

$\Rightarrow \vec{A}\times\vec{B} \perp \vec{A}$ & $\vec{B}$

Need "Back-Cab" formula

$$\vec{A}\times(\vec{B}\times\vec{C}) = \underbrace{\vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})}_{\text{perpendicular to } \vec{A} \text{ \& } \vec{B}\times\vec{C}\ \checkmark}$$

Check x component

$$A_y(B_x C_y - B_y C_x) - A_z(B_z C_x - B_x C_z)$$

$$\stackrel{?}{=} B_x(A_x C_x + A_y C_y + A_z C_z) - C_x(A_x B_x + A_y B_y + A_z B_z)$$

Claim the coordinates of $\vec A \times \vec B$

are $(\vec A \times \vec B)_x = A_y B_z - A_z B_y$

$$(\vec A \times \vec B)_y = A_z B_x - A_x B_z$$

$$(\vec A \times \vec B)_z = A_x B_y - A_y B_x$$

note $\vec C \cdot (\vec A \times \vec B) = \det \begin{pmatrix} C_x & C_y & C_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{pmatrix}$

just the dot product

$$= C_x(A_y B_z - A_z B_y) + C_y(A_z B_x - A_x B_z) + C_z(A_x B_y - A_y B_x)$$

$\Rightarrow \vec A \times \vec B \perp \vec A$ & $\vec B$

Need "Back-Cab" formula

$$A \times (\vec B \times \vec C) = \underbrace{\vec B(\vec A \cdot \vec C) - \vec C(A \cdot B)}_{\text{perpendicular to } \vec A \,\&\, \vec B \times \vec C} \checkmark$$

check x component

$$A_y(B_x C_y - B_y C_x) - A_z(B_z C_x - B_x C_z)$$

$$\stackrel{?}{=} B_x(A_x C_x + A_y C_y + A_z C_z) - C_x(A_x B_x + A_y B_y + A_z C_z)$$

Finally check

$$|\vec A \times \vec B|^2 = (\vec A \times \vec B) \cdot (\vec A \times \vec B)$$

$$= \vec A \cdot (\vec B \times (\vec A \times \vec B))$$

$$= \vec A \cdot (\vec A(\vec B \cdot \vec B) - \vec B(A \cdot \vec B))$$

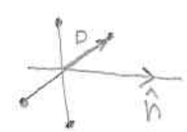
$$= |A|^2 |B|^2 - |(A \cdot B)|^2 \checkmark$$

$$= |A|^2 |B|^2 (1 - \cos^2\theta) = |A|^2 |B|^2 \sin^2\theta$$

Thus, if $\vec F(\vec r)$ is a vector so is $\vec\nabla \times \vec F$.

Can we understand curl?

Use Purcell's curl meter



Let $\vec F(\vec r)$ act on all four masses

$D^2 \hat n \cdot (\vec\nabla \times \vec F)$ = torque $\vec F$ exerts about $\hat n$.

Return later after studying torque

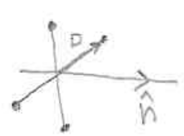
Finally check

$$|\vec A\times\vec B|^2 = (\vec A\times\vec B)\cdot(\vec A\times\vec B)$$
$$= \vec A\cdot(\vec B\times(\vec A\times\vec B))$$
$$= \vec A\cdot(\vec A(\vec B\cdot\vec B) - \vec B(A\cdot\vec B))$$
$$= |A|^2|B|^2 - |(A\cdot B)|^2 \quad \checkmark$$
$$= |A|^2|B|^2(1-\cos^2\theta) = |A|^2|B|^2\sin^2\theta$$

Thus, if $\vec F(\vec r)$ is a vector so is $\vec\nabla\times\vec F$.

c) Can we understand curl?

Use Purcell's curl meter



Let $\vec F(\vec r)$ act on all four masses

$D^2\,\hat n\cdot(\vec\nabla\times\vec F)$ = torque $\vec F$ exerts about $\hat n$

True for small $\vec D$

Return later after studying torque

d) Examples

i) If $\vec F(\vec r) = \hat r f(r)$, find $\vec\nabla\times\vec F$

(we must get $\vec 0$ since $\vec F$ is conservative)

$$\vec\nabla\times\left(\vec r\,\frac{f(r)}{r}\right) = (\vec\nabla\times\vec r)\frac{f(r)}{r} - \vec r\times\underbrace{\left(\vec\nabla\,\frac{f(r)}{r}\right)}_{\hat r\,\frac{\partial}{\partial r}\left(\frac{f(r)}{r}\right)}$$

examine $(\vec\nabla\times\vec r)_x$

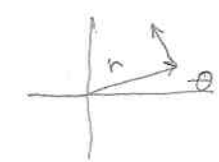
$$= \frac{\partial}{\partial y}z - \frac{\partial}{\partial z}y = 0$$

& $\vec r\times\hat r = 0$

$$\Rightarrow \vec\nabla\times(\hat r\, f(r)) = 0$$

ii) 2-dim problem: $\vec F(\vec r) = \hat\theta\, f(r)\, r$

$$F_x = -r\sin\theta\, f(r)$$
$$F_y = r\cos\theta\, f(r)$$



$$(\vec\nabla\times\vec F)_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{\partial}{\partial x}(x f(r)) + \frac{\partial}{\partial y}(y f(r))$$
$$= 2f(r) + \frac{x^2+y^2}{r}f'(r)$$
$$= 2f(r) + r f'(r) \neq 0$$

d) Examples

i) If $\vec F(\vec r) = \hat r\, f(r)$, find $\vec\nabla\times\vec F$

(we must get $\vec 0$ since $\vec F$ is conservative)

$$\vec\nabla\times\left(\vec r\,\frac{f(r)}{r}\right) = (\vec\nabla\times\vec r)\,\frac{f(r)}{r} - \vec r\times\underbrace{\left(\vec\nabla\,\frac{f(r)}{r}\right)}_{\hat r\,\frac{\partial}{\partial r}\left(\frac{f(r)}{r}\right)}$$

examine $(\vec\nabla\times\vec r)_x$

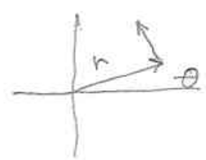
$$= \frac{\partial}{\partial y} z - \frac{\partial}{\partial z} y = 0 \qquad \& \ \vec r\times\hat r = 0$$

$$\Rightarrow \quad \vec\nabla\times\left(\hat r\, f(r)\right) = 0$$

ii) 2-dim problem: $\vec F(\vec r) = \hat\theta\, f(r)\, r$

$$F_x = -r\sin\theta\, f(r)$$

$$F_y = r\cos\theta\, f(r)$$



$$(\vec\nabla\times\vec F)_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{\partial}{\partial x}\left(x f(r)\right) + \frac{\partial}{\partial y}\left(y f(r)\right)$$

$$= 2f(r) + \frac{x^2+y^2}{r} f'(r)$$

$$= 2f(r) + r f'(r) \neq 0$$

Unless $f'(r) = -\frac{2}{r} f(r)$

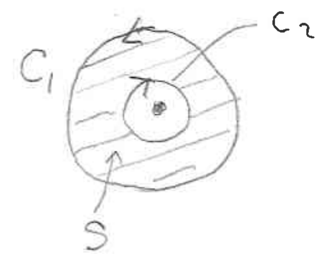
solved by $f(r) = \frac{c}{r^2}$,

recall $\vec F = r\hat\theta f(r) = \hat\theta\,\frac{c}{r}$

$$\Rightarrow \vec\nabla\times\vec F = 0 \quad \text{but} \quad \oint \vec F(\vec r)\cdot d\vec r = \oint \frac{c}{r}\,\hat\theta\cdot d\vec r$$

$$= \int_0^{2\pi} \frac{c}{r}\, r\, d\theta = 2\pi c \overset{?}{\neq} 0$$

Stokes' was misapplied $\hat\theta\,\frac{c}{r}$ is singular at $r=0$ so we must remove this point which adds a new boundary



$$\partial S = C_1 \cup C_2$$

$$0 = \int_S (\vec\nabla\times\vec F)\cdot\hat n\, dS$$

$$= \int_{\partial S} \vec F\cdot d\vec r$$

$$= \int_{C_1} \vec F\cdot d\vec r + \int_{C_2} \vec F\cdot d\vec r$$

$$= 2\pi c - 2\pi c = 0 \ ✓$$

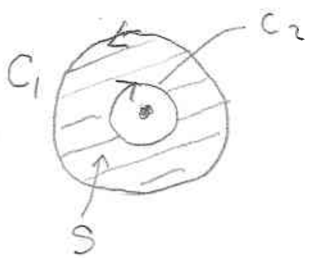
Unless $f'(r) = -\frac{2}{r} f(r)$ recall $\vec{F} = r\hat{\theta} f(r) = \hat{\theta}\,\frac{c}{r}$.

solved by $f(r) = \frac{c}{r^2}$,

$\Rightarrow \vec{\nabla}\times\vec{F} = 0$ but $\oint \vec{F}(\vec{r})\cdot d\vec{r} = \oint \frac{c}{r}\,\hat{\theta}\cdot dr$

$$= \int_0^{2\pi} \frac{c}{r}\, r\, d\theta = 2\pi c \overset{?}{\neq} 0$$

Stokes' was misapplied $\hat{\theta}\,\frac{c}{r}$ is singular at $r = 0$ so we must remove this point which adds a new boundary



$$\partial S = C_1 \cup C_2$$

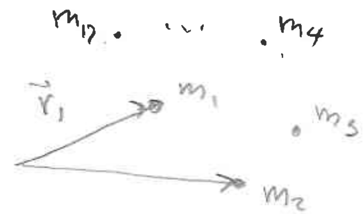
$$0 = \int_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, dS$$

$$= \int_{\partial S} \vec{F}\cdot d\vec{r}$$

$$= \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot dr$$

$$= 2\pi c - 2\pi c = 0 \checkmark$$

5. Energy for a many particle sysetem



Consider a system of N particles at position $\vec{r}_i$ $1 \le i \le N$ and masses m_i

Assume force of 1 on 2, $\vec{F}_{1\,on\,2}(\vec{r}_1, \vec{r}_2)$ is conservative $\vec{F}_{1\,on\,2} = -\vec{\nabla}_{r_2} U(\vec{r}_1, \vec{r}_2)$.

Natural to expect that

$$\vec{F}_{2\,on\,1} = -\vec{\nabla}_{r_2} U(\vec{r}_1, \vec{r}_2)$$

Assume $U(\vec{r}_1, \vec{r}_2)$ does not change if we translate $\vec{r}_1$ & $\vec{r}_2$

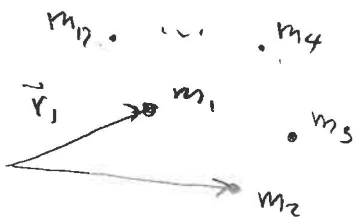
$$U(\vec{r}_1, \vec{r}_2) = U(\vec{r}_1 + \vec{D}, \vec{r}_2 + \vec{D})$$

Choose $\vec{D} = -\vec{r}_2$

$$= U(\vec{r}_1 - \vec{r}_2, 0)$$

$$\equiv U(\vec{r}_1 - \vec{r}_2)$$

5. Energy for a many particle syseten



Consider a system of N particles at position $\vec{r}_i$ $1 \le i \le N$ and masses m_i

Assume force of 1 on 2, $\vec{F}_{1 on 2}(\vec{r}_1, \vec{r}_2)$ is conservative $\vec{F}_{1 on 2} = -\vec{\nabla}_{r_2} U(\vec{r}_1, \vec{r}_2)$.

Natural to expect that

$$\vec{F}_{2 on 1} = -\vec{\nabla}_{r_2} U(\vec{r}_1, \vec{r}_2)$$

Assume $U(\vec{r}_1, \vec{r}_2)$ does not change if we translate $\vec{r}_1$ & $\vec{r}_2$

$$U(\vec{r}_1, \vec{r}_2) = U(\vec{r}_1 + \vec{D}, \vec{r}_2 + \vec{D})$$

Choose $\vec{D} = -\vec{r}_2$

$$= U(\vec{r}_1 - \vec{r}_2, 0)$$

$$\equiv U(\vec{r}_1 - \vec{r}_2)$$

Since $\vec{\nabla}_{r_1} U(\vec{r}_1 - \vec{r}_2) = -\vec{\nabla}_{r_2} U(\vec{r}_1 - \vec{r}_2)$

$$\vec{F}_{2 on 1}(\vec{r}_1 - \vec{r}_2) = -\vec{F}_{1 on 2}(\vec{r}_1 - \vec{r}_2)$$

Newton's 3^{rd} Law $\Rightarrow$ $\dfrac{d\vec{P}_{tot}}{dt} = 0$

Invariance under translation implies momentum conservation.

Energy must also be conserved

$$E = \sum_{i=1}^{N} \frac{1}{2} m_i \left(\frac{d\vec{r}_i}{dt}\right)^2 + \frac{1}{2} \sum_{i \neq j} U_{ij}(\vec{r}_i - \vec{r}_j)$$

Since each particle experiences a sum of conservative forces.

Look at terms which depend on $\vec{r}_i$:

$$\frac{d}{dt}\left\{ \frac{1}{2} m_i \left(\frac{d\vec{r}_i}{dt}\right)^2 + \sum_{\substack{j \\ j \neq i}} U_{ij}(\vec{r}_i - \vec{r}_j) \right\}$$

$$= \left\{ m_i \frac{d^2\vec{r}_i}{dt^2} - \sum_{\substack{j \\ j \neq i}} \vec{F}_{j on i} \right\} \cdot \frac{d\vec{r}_i}{dt} = 0$$