Since $\vec{\nabla}_{r_1} U(\vec{r}_1 - \vec{r}_2) = -\vec{\nabla}_{r_2} U(\vec{r}_1 - \vec{r}_2)$

$$\vec{F}_{2\,on\,1}(\vec{r}_1 - \vec{r}_2) = -\vec{F}_{1\,on\,2}(r_1 - \vec{r}_2)$$

Newton's 3rd Law $\Rightarrow \dfrac{d\vec{P}_{tot}}{dt} = 0$

Invariance under translation implies momentum conservation.

Energy must also be conserved

$$E = \sum_{i=1}^{N} \frac{1}{2} m_i \left(\frac{d\vec{r}_i}{dt}\right)^2 + \frac{1}{2} \sum_{i \neq j} U_{ij}(\vec{r}_i - \vec{r}_j)$$

Since each particle experiences a sum of conservative forces.

Look at terms which depend on $\vec{r}_i$:

$$\frac{d}{dt}\left\{ \frac{1}{2} m_i \left(\frac{dr_i}{dt}\right)^2 + \sum_{\substack{j \\ j \neq i}} U_{ij}(\vec{r}_i - \vec{r}_j) \right\}$$

$$= \left\{ m_i \frac{d^2\vec{r}_i}{dt^2} - \sum_{\substack{j \\ j \neq i}} \vec{F}_{j\,on\,i} \right\} \cdot \frac{d\vec{r}_1}{dt} = 0$$

6) Use energy and momentum conservation to discuss collisions

a) 1-dim before $\overset{m_1\ v_1}{\longrightarrow}$ $\overset{m_2\ v_2}{\longleftarrow}$

after $\underset{v_1'\ m_1}{\longleftarrow}$ $\underset{m_2\ v_2'}{\longrightarrow}$

Energy: $\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2$

Momentum: $m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$

rewrite $m_1(v_1^2 - v_1'^2) = m_2(v_2'^2 - v_2^2)$

$m_1(v_1 - v_1') = m_1(v_2' - v_2)$

divide $v_1 + v_1' = v_2 + v_2'$

eliminate v_2' & determine v_1'

$$m_1(v_1 - v_1') = m_2(v_1 + v_1' - v_2 - v_2)$$

or $v_1' = \dfrac{(m_1 - m_2)v_1 + 2v_2 m_2}{m_1 + m_2}$

Examine lab & center-of-mass systems

6) Use energy and momentum conservation to discuss collisions

a) 1-dim

before: $m_1\ v_1 \longrightarrow \qquad \longleftarrow m_2\ v_2$

after: $\longleftarrow v_1'\ m_1 \qquad m_2\ v_2' \longrightarrow$

Energy: $\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2$

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Examine lab & center-of-mass systems

- Lab system call 2 the target and choose $v_2 = 0$

$v_1' = \dfrac{m_1 - m_2}{m_1 + m_2} v_1$ $\qquad m_1 > m_2 \quad v_1' > 0$

$m_1 < m_2 \quad v_1' < 0$

What happened to the case where the particles do not scatter and $v_1' = v_1$, $v_2' = v_2$?

- Center of mass system

$R_{cm} = \dfrac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$, $\quad v_{cm} = \dfrac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$

$v_1^{cm} = v_1 - v_{cm} = v_1 - \dfrac{m_1 v_1}{m_1 + m_2} = \dfrac{m_2 v_1}{m_1 + m_2}$

$v_2^{cm} = 0 - v_{cm} = -\dfrac{m_1 v_1}{m_1 + m_2}$

$p_1^{cm} = m_1 v_1^{cm} = \dfrac{m_1 m_2 v_1}{m_1 + m_2}$

$p_2^{cm} = m_2 v_2^{cm} = -\dfrac{m_1 m_2 v_1}{m_1 + m_2} = -p_1^{cm}$

- Lab system call 2 the target and choose $v_2 = 0$

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$$p_2^{cm} = m_2 v_2^{cm} = -\frac{m_1 m_2 v_1}{m_1 + m_2} = -p_1^{cm}$$

$$p_{tot}^{cm} = p_1^{cm} + p_2^{cm} = (m_1 + m_2) v_{cm}^{cm} = 0$$

In the CM system where $v_{cm} = 0$ we must have $p_1^{cm} = -p_2^{cm}$!

In CM system collisions are simple

$$\underset{\text{before}}{\overrightarrow{p_1^{cm}} \;\; \overleftarrow{p_2^{cm}}} \quad \Rightarrow \quad \underset{\text{after}}{\overleftarrow{-p_1^{cm}} \;\; \overrightarrow{-p_2^{cm}}}$$

Easy to see $p_1'^{cm} = -p_2'^{cm}$ since p_{tot}^{cm} must still be 0

While energy conservation requires

$$\frac{(p_1^{cm})^2}{2m_1} + \frac{(p_2^{cm})^2}{2m_2} = \frac{(p_1'^{cm})^2}{2m_1} + \frac{(p_2'^{cm})^2}{2m_2}$$

$$\Rightarrow \left(\frac{1}{2m_1} + \frac{1}{2m_2}\right)(p_1^{cm})^2 = \left(\frac{1}{2m_1} + \frac{1}{2m_2}\right)(p_1'^{cm})^2$$

$$\Rightarrow \quad p_1^{cm} = \pm p_1'^{cm}$$

Go back to lab $v_1' = v_1'^{cm} + v_{cm}$

$$v_1' = -\frac{m_2 v_1}{m_1 + m_2} + \frac{m_1 v_1}{m_1 + m_2} = \frac{m_1 - m_2}{m_1 + m_2} v_1 \quad ✓$$

$$P_{tot}^{cm} = p_1^{cm} + p_2^{cm} = (m_1 + m_2)\,v_{cm}^{cm} = 0$$

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In CM system collisions are simple

$$\xrightarrow{p_1^{cm}} \ \xleftarrow{p_2^{cm}} \quad \Rightarrow \quad \xleftarrow{-p_1^{cm}} \ \xrightarrow{-p_2^{cm}}$$

before after

Easy to see $p_1'^{cm} = -p_2'^{cm}$

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While energy conservation requires

$$\frac{(p_1^{cm})^2}{2m_1} + \frac{(p_2^{cm})^2}{2m_2} = \frac{(p_1'^{cm})^2}{2m_1} + \frac{(p_2'^{cm})^2}{2m_2}$$

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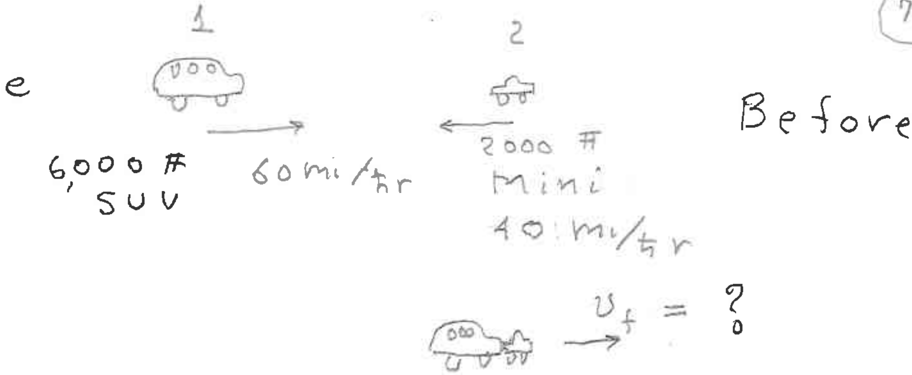
Example



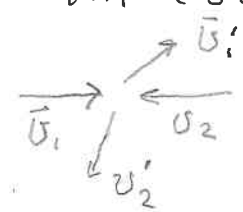
Use momentum conservation

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2)\,v_f$$

$$v_f = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = \frac{6{,}000 \times 60 - 2{,}000 \times 40}{6{,}000 + 2{,}000}\ \frac{mi}{hr}$$

$$= 35\ \frac{mi}{hr}$$ SUV is slowed a little

b) 3-dim collisions



Assume $\vec v_1$ & $\vec v_2$ are parallel.

$\vec v_1'$ & $\vec v_1$ or $\vec v_2$ define a plane

and momentum conservation

$$m_2 \vec v_2' = m_1 \vec v_1 + m_2 \vec v_2 - m_1 \vec v_1'$$

implies $\vec v_2'$ lies in that plane.

Example

1 2

6,000 # SUV 60 mi/hr

2000 # mini 40 mi/hr

Before

$v_f = ?$

Use momentum conservation

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) v_f$$

$$v_f = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = \frac{6{,}000 \times 60 - 2{,}000 \times 40}{6{,}000 + 2{,}000} \frac{mi}{hr}$$

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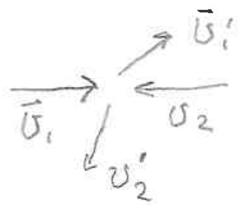

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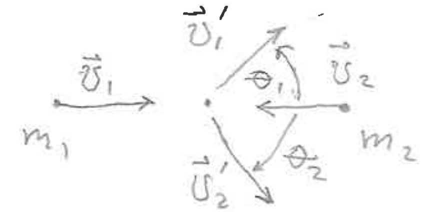
Count variables and equations



Usually $|\vec{v}_1|$ & $|\vec{v}_2|$ are given.

$|\vec{v}_1'|$, $|\vec{v}_2'|$, θ_1 & θ_2 are unknown

3 Equations

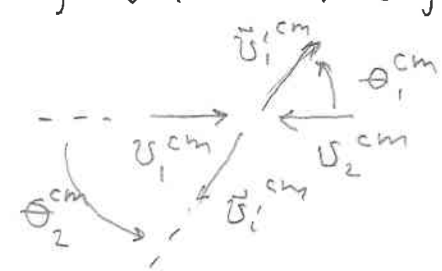
$$m_1 \vec{v}_1 + m_2 \vec{v}_2 = m_1 \vec{v}_1' + m_2 \vec{v}_2' \quad \} \text{ 2-dim}$$

$$\frac{1}{2} m_1 \vec{v}_1^{\,2} + \frac{1}{2} m_2 \vec{v}_2^{\,2} = \frac{1}{2} m_1 \vec{v}_1'^{\,2} + \frac{1}{2} m_2 \vec{v}_2'^{\,2}$$

$4 - 3 = 1$ unknown: measure θ_1

and θ_2, v_1' & v_2' are all known!

Easy in CM system:



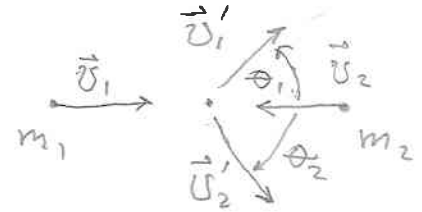
$$m_1 v_1^{cm} = m_2 v_2^{cm}$$

$$v_1^{cm} = v_1'^{cm}$$

$$v_2^{cm} = v_2'^{cm}$$

$$\theta_1^{cm} = \theta_2^{cm}$$

Count variables and equations



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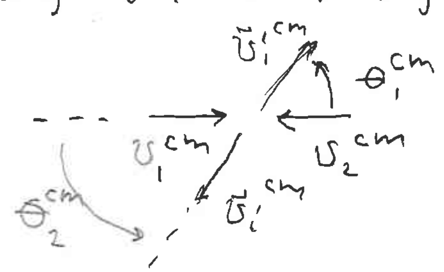

$$m_1 v_1^{cm} = m_2 v_2^{cm}$$

$$v_1^{cm} = v_1'^{cm}$$

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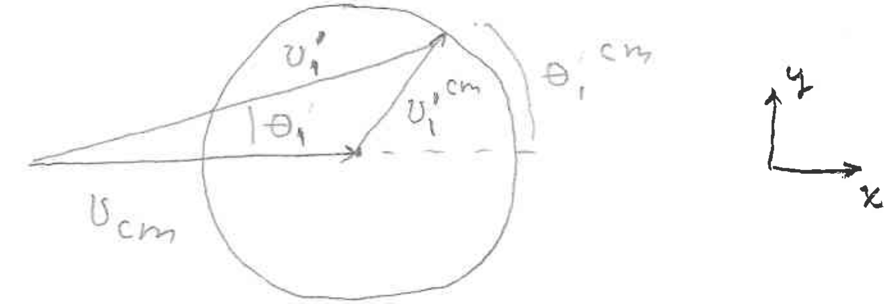
$$\theta_1^{cm} = \theta_2^{cm}$$

Use cm geometry to find $v_1'(\theta_1)$ in lab



Equate y components of v_1'

$$v_1' \sin\theta_1 = v_1'^{cm} \sin\theta_1^{cm}$$

Relate x components of v_1'

$$v_1' \cos\theta_1 = v_{cm} + v_1'^{cm} \cos\theta_1^{cm}$$

Combine to eliminate θ_1^{cm}, use $v_1^{cm} = v_1'^{cm}$

$$v_1'^2 \sin^2\theta_1 + (v_1' \cos\theta_1 - v_{cm})^2 = v_1^{cm\,2}$$

$$v_1' - 2 v_1' \cos\theta_1 \underbrace{\frac{m_1 v_1}{m_1+m_2}}_{v_{cm}} + \Big(\underbrace{\frac{m_1 v_1}{m_1+m_2}}_{v_{cm}}\Big)^2 = \Big(\underbrace{\frac{m_2 v_1}{m_1+m_2}}_{v_1^{cm}}\Big)^2$$

Use cm geometry to find

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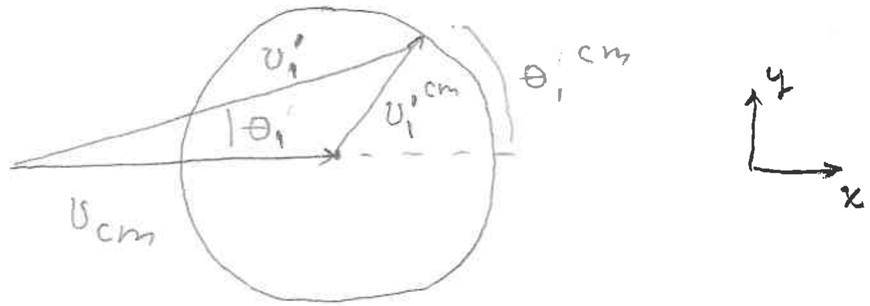

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Combine to eliminate θ_1^{cm},

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$$v_1'^2 \sin^2\theta_1' + (v_1' \cos\theta_1 - v_{cm})^2 = v_1^{cm\,2}$$

$$v_1' - 2 v_1' \cos\theta_1 \underbrace{\frac{m_1 v_1}{m_1+m_2}}_{v_{cm}} + \underbrace{\left(\frac{m_1 v_1}{m_1+m_2}\right)^2}_{v_{cm}} = \underbrace{\left(\frac{m_2 v_1}{m_1+m_2}\right)^2}_{v_1^{cm}}$$

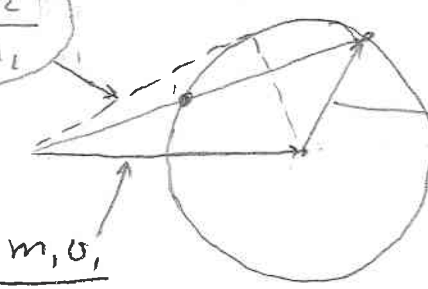
This is a quadratic equation giving $v_1'(\theta)$

$$v_1' = \frac{1}{2}\left\{ \frac{2m_1 v_1}{m_1+m_2}\cos\theta \pm \sqrt{\left(\frac{2m_1 v_1}{m_1+m_2}\cos\theta\right)^2 - 4\left(\frac{m_1^2 - m_2^2}{m_1^2+m_2^2}\right) v_1^2} \right\}$$

$$= \frac{v_1}{m_1+m_2}\left\{ m_1 \cos\theta \pm \sqrt{m_1^2 \cos^2\theta - m_1^2 + m_2^2} \right\}$$

$$= \frac{v_1}{m_1+m_2}\left\{ m_1 \cos\theta \pm \sqrt{m_2^2 - m_1^2 \sin^2\theta} \right\}$$

$\sin\theta_{max} = \dfrac{m_2}{m_1}$



$v_1^{cm} = \dfrac{m_2 v_1}{m_1+m_2}$

$v_{cm} = \dfrac{m_1 v_1}{m_1+m_2}$

Diagram shows

$m_1 > m_2$

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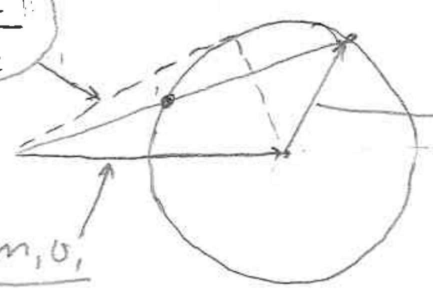

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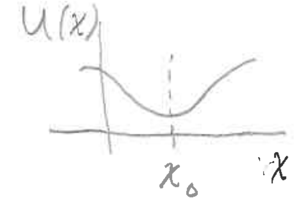
E Simple harmonic motion

1\.



Takes place whenever a particle moves near the minimum of the potential

$$U(x) = U(x_0) + \cancel{\frac{dU}{dx}}^{\,0}(x-x_0) + \frac{1}{2}\frac{d^2U}{dx^2}(x_0)(x-x_0)^2 + \cancel{\frac{1}{3!}\frac{d^3U}{dx^3}(x-x_0)^3} + \ldots$$

neglect anharmonic terms

Choose $x_0 = 0$, $U(x_0) = 0$, let $k = \frac{d^2U}{dx^2}(x_0)$

$$U(x) = \frac{1}{2}kx^2$$

$$F(x) = -\frac{dU}{dx} = -kx$$

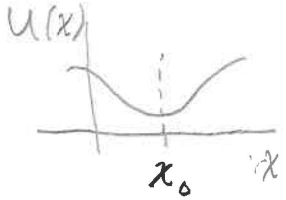
a linear restoring force

"Hooke's Law"

Newton's law $m\frac{d^2x}{dt^2} = -kx$

is easy to solve.

E Simple harmonic motion

1. 

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$$U(x) = U(x_0) + \cancelto{0}{\frac{dU}{dx}(x-x_0)} + \frac{1}{2}\frac{d^2U}{dx^2}(x_0)\,(x-x_0)^2$$

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Newton's law $m\frac{d^2x}{dt^2} = -kx$

is easy to solve.

$$\frac{d}{dt}\cos\omega t = -\omega\sin\omega t \qquad \frac{d}{dt}\sin\omega t = \omega\cos\omega t$$

Thus $x(t) = A\cos(\omega t + \phi)$

where $\omega^2 = \frac{k}{m}$ and A & ϕ are determined by the particle's initial position and velocity

$$x(0) = A\cos\phi$$

$$\dot{x}(0) = -A\omega\sin\phi$$

$$A = \sqrt{x(0)^2 + \frac{1}{\omega^2}\dot{x}(0)^2} \qquad \tan\phi = \frac{-\dot{x}(0)}{\omega x(0)}$$

- $|x(t)| \le |A|$ so if A is small the particle will stay with in the region where anharmonic terms in $U(x)$ can be neglected