$$\frac{d}{dt}\cos\omega t = -\omega \sin\omega t \qquad \frac{d}{dt}\sin\omega t = \omega\cos\omega t$$

Thus $x(t) = A\cos(\omega t + \phi)$

where $\omega^2 = \frac{k}{m}$ and A & ϕ are determined by the particle's initial position and velocity

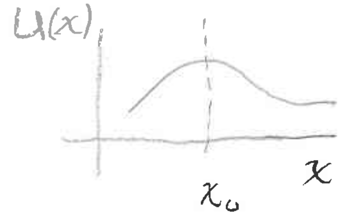
$$x(0) = A\cos\phi$$

$$\dot{x}(0) = -A\omega\sin\phi$$

$$A = \sqrt{x(0)^2 + \frac{1}{\omega^2}\dot{x}(0)^2} \qquad \tan\phi = \frac{-\dot{x}(0)}{\omega x(0)}$$

- $|x(t)| \le |A|$ so if A is small the particle will stay with in the region where anharmonic terms in $U(x)$ can be neglected

- Not true for related motion about a maximum of $U(x)$:



$$U(x) = U(x_0) + \cancel{\frac{dU}{dx}(x-x_0)}^{\,0}$$

$$+ \frac{1}{2}\underbrace{\frac{d^2U}{dx^2}}_{<0}(x-x_0)^2$$

Now we can choose

$x_0 = 0$, $U(x_0) = 0$, $k = -\frac{d^2U}{dx^2}(x_0) > 0$

$$U(x) = -\frac{1}{2}kx^2 \quad \& \quad m\frac{d^2x}{dt^2} = +kx$$

The opposite of a restoring force!

$$x(t) = a e^{+\sqrt{k/m}\,t} + b e^{-\sqrt{k/m}\,t}$$

or $= A\cosh(\sqrt{k/m}\,t + \phi)$

Unless we can miraculously choose $a = 0$ exactly the particle will quickly leave the "harmonic" region

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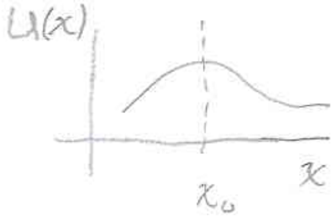

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Now we can choose

$x_0 = 0$, $U(x_0) = 0$, $\kappa = -\frac{d^2U}{dx^2}(x_0) > 0$

$$U(x) = -\frac{1}{2}\kappa x^2 \Rightarrow m\frac{d^2x}{dt^2} = +\kappa x$$

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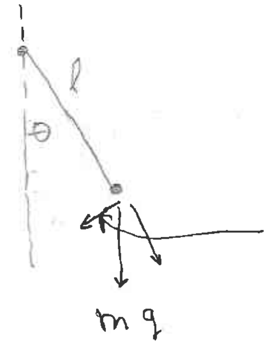
2. Consider pendulum motion

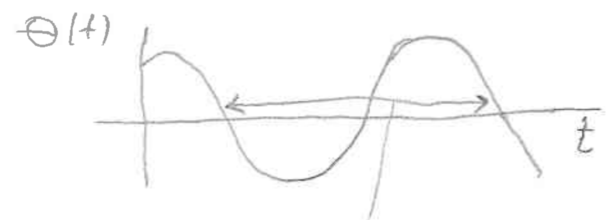


$$m l\ddot{\theta} = -mg\underbrace{\sin\theta}_{\theta - \frac{1}{3!}\theta^3 + \dots}$$

or $\ddot{\theta} \simeq -\frac{g}{l}\theta$

again oscillating about the minimum of the energy at $\theta = 0$

$\theta(t) = A\cos(\omega t + \phi)$ $\quad \omega = \sqrt{\frac{g}{l}}$



period T, $T\omega = 2\pi$, $T = \frac{2\pi}{\omega}$

Frequency ν is the number of cycles per unit time so

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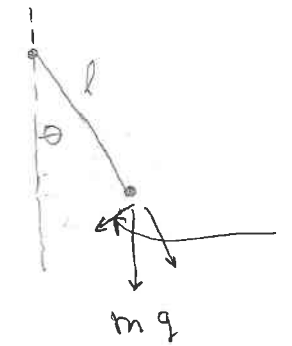

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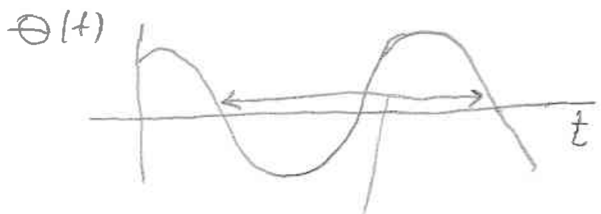

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3. Simple harmonic motion with Complex numbers.

Our equation $m\frac{d^2x}{dt^2} + kx = 0$

is an example of a linear ordinary differential equation with constant coefficients which should be easy to solve with a simple exponential:

$$x(t) = A e^{\lambda t}$$

$$m\lambda^2 \cancel{A e^{\lambda t}} + k\cancel{A e^{\lambda t}} = 0$$

Requires λ to be a root of the polynomial

$$m\lambda^2 + k = 0$$

an equation that can be solved only if we introduce complex numbers $\lambda = \pm\sqrt{-\frac{k}{m}} = \pm i\sqrt{\frac{k}{m}}$

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Expand the real numbers by including an extra element $i = \sqrt{-1}$.

A general complex number z has the form $z = a + ib$

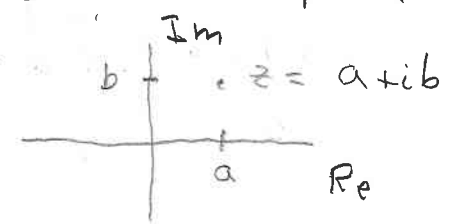
where for $z_1 = a_1 + i b_1$ & $z_2 = a_2 + i b_2$

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$

and for real r, $r \cdot z = (ra) + i(rb)$

Thus, the complex numbers form a real, two-dimensional vector space

$$z = a + ib \equiv (a, b)$$



However, we can also multiply z_1 & z_2

$$z_1 \cdot z_2 = (a_1 + ib_1)(a_2 + ib_2)$$

$$= a_1 a_2 - b_1 b_2 + i(b_1 a_2 + a_1 b_2)$$

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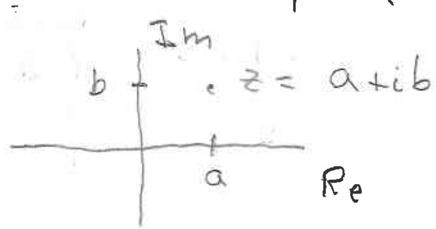

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With this multiplication law, the complex numbers form a field

$0 + i \cdot 0$ is the zero element

$1 + i \cdot 0$ is the unit element

$$0 + z = z, \quad 1 \cdot z = z, \quad 0 \cdot z = 0$$

and if $z \neq 0$, we can find $\frac{1}{z}$ obeying $z \cdot \frac{1}{z} = 1$ as follows:

if $z = a + ib$, define $z^* = a - ib$, the "complex conjugate of z

Then $z z^* = (a+ib)(a-ib) = a^2 + b^2$

use $\frac{1}{z} = \frac{z^*}{z z^*}$

Other requirements of a field

$$z_1 \cdot z_2 = z_2 \cdot z_1 \qquad z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$$

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If $z = a+ib$

define $re(z) \equiv a = \frac{z+z^*}{2}$

$im(z) \equiv b = \frac{z-z^*}{2i}$

Return to $m\frac{d^2x}{dt^2} = -kx$ and $x(t) = A$

and $x(t) = A \underbrace{e^{\pm i\sqrt{\frac{k}{m}}t}}_{?}$

Define $e^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{3!}z^3 + \cdots$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

This must obey all of the properties of exponents of real numbers since both real and complex numbers add and multiply following the same rules

$$e^{z_1} e^{z_2} = e^{z_1+z_2}, \qquad \frac{1}{e^z} = e^{-z}$$

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So we need to figure out how to determine e^{ib}:

$e^{ib} = \sum_{n=0}^{\infty} \frac{1}{n!}(ib)^n$ separate into even and odd powers

$$= \underbrace{\sum_{n=0}^{\infty} \frac{1}{(2n)!}(-1)^n b^{2n}}_{\text{even powers}} + i \underbrace{\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(-1)^n b^{2n+1}}_{\text{odd powers}}$$

$= \cos b$ $= \sin b$

or $e^{ib} = \cos b + i \sin b$

$|z|^2 = e^{ib}e^{-ib} = \cos^2 b + \sin^2 b = 1$

$e^{i(\theta + \phi)} = e^{i\theta} e^{i\phi}$

$\cos(\theta+\phi) + i\sin(\theta+\phi) = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$

$= \underbrace{\cos\theta\cos\phi - \sin\theta\sin\phi}_{\cos(\theta+\phi)}$ ✓

$+ i(\underbrace{\sin\theta\cos\phi + \cos\theta\sin\phi}_{\sin(\theta+\phi)})$ ✓

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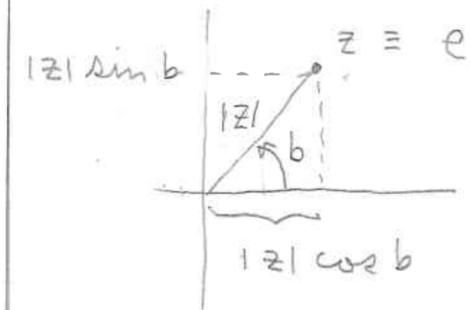
Viewed in 2-dim plot



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since $|z|^2 = zz^* = e^{2a}$

$$z_1 z_2 = |z_1| e^{i\theta_1} |z_2| e^{i\theta_2}$$

$$= |z_1||z_2| e^{i(\theta_1+\theta_2)}$$

"multiply the magnitudes and add the angles"

4. Apply to damped simple harmonic motion.

Introduce a viscous or damping force proportion to the velocity

$$F_{fric} = -\sigma \frac{dx}{dt}$$

so

$$m\frac{d^2x}{dt^2} = -\sigma\frac{dx}{dt} - kx$$ easily solved with complex numbers

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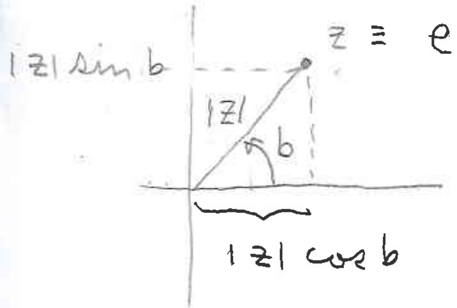

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Start with

$$m \frac{d^2x}{dt^2} + \sigma \frac{dx}{dt} + kx = 0 \qquad \text{try } x(t) = Ae^{\lambda t}$$

$$m\lambda^2 \cancel{Ae^{\lambda t}} + \sigma \lambda \cancel{Ae^{\lambda t}} + k \cancel{Ae^{\lambda t}} = 0$$

or

$$m\lambda^2 + \sigma\lambda + k = 0$$

$$\lambda^2 + \underbrace{\frac{\sigma}{m}}_{\gamma} \lambda + \underbrace{\frac{k}{m}}_{\omega_0^2} = 0$$

$$\lambda = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2} = -\frac{\gamma}{2} \pm \sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}$$

Thus a general solution is

$$x(t) = A e^{-\frac{\gamma}{2}t} e^{+\sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}\, t} + B e^{-\frac{\gamma}{2}t} e^{-\sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}\, t}$$

- Since the equation is linear in $x(t)$ we can add two solutions and will still have a solution.
- While A & B may be complex since the equation has real coefficients, $x^*(t)$ will be a solution and $x(t) + x^*(t)$ a real solution