Start with

$$m\frac{d^2x}{dt^2} + \sigma\frac{dx}{dt} + kx = 0 \qquad \text{try } x(t) = Ae^{\lambda t}$$

$$m\lambda^2 \cancel{Ae^{\lambda t}} + \sigma\lambda \cancel{Ae^{\lambda t}} + k\cancel{Ae^{\lambda t}} = 0$$

or $m\lambda^2 + \sigma\lambda + k = 0$

$$\lambda^2 + \underbrace{\frac{\sigma}{m}}_{\gamma}\lambda + \underbrace{\frac{k}{m}}_{\omega_0^2} = 0$$

$$\lambda = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2} = -\frac{\gamma}{2} \pm \sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}$$

Thus a general solution is

$$x(t) = Ae^{-\frac{\gamma}{2}t}e^{+\sqrt{\left(\frac{\gamma}{2}\right)^2-\omega_0^2}\,t} + Be^{-\frac{\gamma}{2}t}e^{-\sqrt{\left(\frac{\gamma}{2}\right)^2-\omega_0^2}\,t}$$

- Since the equation is linear in $x(t)$ we can add two solutions and will still have a solution.
- While A & B may be complex since the equation has real coefficients, $x^*(t)$ will be a solution and $x(t)+x^*(t)$ a real solution

There a three types of solution

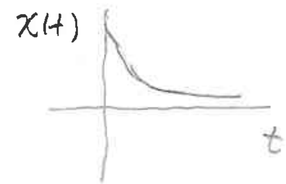
1. Over damped: $\frac{\gamma}{2} > \omega_0$, $\sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}$ real

and we have two real exponents

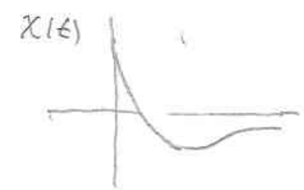
with $\lambda_\pm = -\frac{\gamma}{2} \pm \sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2} < 0$

$$x(t) = A_+ e^{\lambda_+ t} + A_- e^{\lambda_- t}$$

Choose A_+ & A_- to give needed $x(0)$ & $\frac{dx}{dt}(0)$



A_+ & A_- positive



A_+ negative & A_- positive

2. Under damped $\omega_0 > \frac{\gamma}{2}$

$$\sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2} = i\sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2} = i\omega$$

is imaginary

$$x(t) = A_+ e^{-\frac{\gamma}{2}t} e^{+i\omega t} + A_- e^{-\frac{\gamma}{2}t} e^{-i\omega t}$$

There a three types of solution

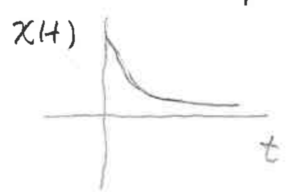
1. Over damped: $\frac{\gamma}{2} > \omega_0$, $\sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}$ real

and we have two real exponents

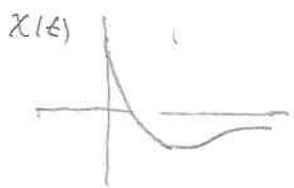
with $\lambda_\pm = -\frac{\gamma}{2} \pm \sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2} < 0$

$$x(t) = A_+ e^{\lambda_+ t} + A_- e^{\lambda_- t}$$

Choose A_+ & A_- to give needed $x(0)$ & $\frac{dx}{dt}(0)$



A_+ & A_- positive



A_+ negative & A_- positive

2. Under damped $\omega_0 > \frac{\gamma}{2}$

$$\sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2} = i\sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2} = i\omega$$

is imaginary

$$x(t) = A_+ e^{-\frac{\gamma}{2}t} e^{+i\omega t} + A_- e^{-\frac{\gamma}{2}t} e^{-i\omega t}$$

now we must choose A_+ & A_- complex to make $x(t)$ real. $A_- = A_+^*$ is needed. Use $A_\pm = \frac{A}{2} e^{i\phi}$ real

$$x(t) = \frac{A}{2} e^{-\frac{\gamma}{2}t}\left(e^{i\omega t} e^{i\phi} + e^{-i\omega t} e^{-i\phi}\right)$$

$$= A e^{-\gamma t} \cos(\omega t + \phi)$$

amplitude decreases as energy is lost to friction

3. Critically damped $\frac{\gamma}{2} = \omega_0$, $\pm\sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2} = 0$

only one obvious solution

$$x(t) = A e^{-\frac{\gamma}{2}t} \ ?$$

construct a second:

$$\lim_{\omega_0 \to \frac{\gamma}{2}} \frac{e^{-\frac{\gamma}{2}t} e^{+\sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}\, t} - e^{-\frac{\gamma}{2}t} e^{-\sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}\, t}}{\sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}}$$

now we must choose A_+ & A_- complex to make $x(t)$ real. $A_- = A_+^*$ is needed. Use $A_\pm = \frac{A}{2} e^{i\phi}$ real

$$x(t) = \frac{A}{2} e^{-\frac{\gamma}{2}t}\left(e^{i\omega t} e^{i\phi} + e^{-i\omega t} e^{-i\phi}\right)$$

$$= A e^{-\gamma t} \cos(\omega t + \phi)$$

amplitude decreases as energy is lost to friction

③ Critically damped $\frac{\gamma}{2} = \omega_0$, $\pm\sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2} = 0$

only one obvious solution

$$x(t) = A e^{-\frac{\gamma}{2}t} \quad ?$$

construct a second:

$$\lim_{\omega_0 \to \frac{\gamma}{2}} \frac{e^{-\frac{\gamma}{2}t} e^{+\sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}\, t} - e^{-\frac{\gamma}{2}t} e^{-\sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}\, t}}{\sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}}$$

$$= e^{-\frac{\gamma}{2}t} \left\{ \frac{\left(1 + \sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}\, t\right) - \left(1 - \sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}\, t\right)}{\sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}} \right\}$$

$$= 2 t e^{-\frac{\gamma}{2}t}$$

So $x(t) = e^{-\frac{\gamma}{2}t}(A + Bt)$

and we still have two solutions and the freedom to choose $x(0)$ & $\frac{dx}{dt}(0)$

note $\frac{d^2}{dt^2} + \gamma \frac{d}{dt} + \frac{\gamma^2}{4} = \left(\frac{d}{dt} + \frac{\gamma}{2}\right)^2$

& $\left(\frac{d}{dt} + \frac{\gamma}{2}\right)^2 t e^{-\gamma t} = 0$

Examine energy loss from damping

assume $\frac{\gamma}{2} \ll \omega$

use $x(t) = A e^{-\gamma t/2} \cos(\omega t + \phi)$

$$E = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + \frac{1}{2} k x^2$$

$$= e^{-\frac{\gamma}{2}t}\left\{\frac{\left(1+\sqrt{\left(\frac{\gamma}{2}\right)^2-\omega_0^2}\,t\right)-\left(1-\sqrt{\left(\frac{\gamma}{2}\right)^2-\omega_0^2}\,t\right)}{\sqrt{\left(\frac{\gamma}{2}\right)^2-\omega_0^2}}\right\}$$

$$= 2te^{-\frac{\gamma}{2}t}$$

So $x(t) = e^{-\frac{\gamma}{2}t}(A+Bt)$

and we still have two solutions and the freedom to choose $x(0)$ & $\frac{dx}{dt}(0)$

note $\frac{d^2}{dt^2}+\gamma\frac{d}{dt}+\frac{\gamma^2}{4} = \left(\frac{d}{dt}+\frac{\gamma}{2}\right)^2$

& $\left(\frac{d}{dt}+\frac{\gamma}{2}\right)^2 te^{-\gamma t} = 0$

Examine energy loss from damping

assume $\frac{\gamma}{2} \ll \omega$

use $x(t) = Ae^{-\gamma t/2}\cos(\omega t+\phi)$

$$E = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2$$

$$E = \frac{1}{2}mA^2e^{-\gamma t}\left(\cancel{-\frac{\gamma}{2}\cos(\omega t+\phi)} - \omega\sin(\omega t+\phi)\right)^2 + \frac{1}{2}kA^2e^{-\gamma t}\cos^2(\omega t+\phi)$$

$$\approx \frac{1}{2}kA^2e^{-\gamma t}$$

Energy loss in one period

$$\Delta E = \frac{1}{2}kA^2\left[e^{-\gamma t} - \underbrace{e^{-\gamma\left(t+\frac{2\pi}{\omega}\right)}}_{e^{-\gamma t}\underbrace{e^{-2\pi\frac{\gamma}{\omega}}}_{\approx 1-2\pi\frac{\gamma}{\omega}}}\right]$$

$$\Delta E = \frac{1}{2}kA^2e^{-\gamma t}\frac{2\pi\gamma}{\omega}$$

$$= E\frac{2\pi\gamma}{\omega}$$

Define the "Q" or quality factor of the ossilator as inverse of fraction of energy lost per radian

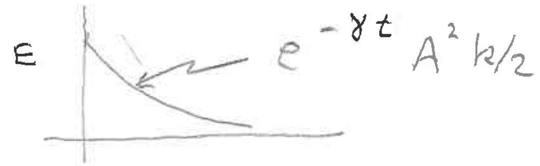
$$Q = \frac{E}{\Delta E/2\pi} = \frac{\omega}{\gamma}$$

The less loss the Larger Q

$$E = \frac{1}{2} m A^2 e^{-\gamma t}\left(\cancel{-\frac{\gamma}{2}\cos(\omega t+\phi)} - \omega \sin(\omega t + \phi)\right)^2$$

$$+ \frac{1}{2} k A^2 e^{-\gamma t} \cos^2(\omega t + \phi)$$

$$\approx \frac{1}{2} k A^2 e^{-\gamma t}$$



Energy loss in one period

$$\Delta E = \frac{1}{2} k A^2 \left[e^{-\gamma t} - \underbrace{e^{-\gamma\left(t + \frac{2\pi}{\omega}\right)}}_{e^{-\gamma t}\, \underbrace{e^{-2\pi \frac{\gamma}{\omega}}}_{\approx 1 - 2\pi \frac{\gamma}{\omega}}} \right]$$

$$\Delta E = \frac{1}{2} k A^2 e^{-\gamma t} \frac{2\pi\gamma}{\omega}$$

$$= E \frac{2\pi\gamma}{\omega}$$

Define the "Q" or quality factor of the ossilator as inverse of fraction of energy lost per radian

$$Q = \frac{E}{\Delta E / 2\pi} = \frac{\omega}{\gamma}$$

The less loss the Larger Q

Calculate directly the negative work done by friction in one period

$$\Delta E = -\oint F \cdot dx = \oint \sigma \frac{dx}{dt}\, dx$$

$$= m\gamma \int_t^{t+T} \left(\frac{dx}{dt}\right)^2 dt$$

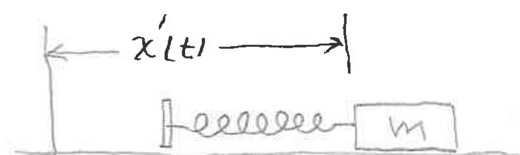
$$\approx m\gamma\, \omega^2 \int_t^{t+T} A^2 e^{-\gamma t} \sin^2(\omega t + \phi)\, dt$$

$$\approx \gamma k A^2 e^{-\gamma t} \underbrace{\int_t^{t+T} \sin^2(\omega t + \phi)\, dt}_{1/2 \cdot T}$$

$$\Delta E = \gamma k A^2 e^{-\gamma t} \frac{1}{2} \frac{2\pi}{\omega} = 2\pi \frac{\gamma}{\omega} \cdot E \quad ✓$$

5. Forced simple harmonic motion

Add a harmonic, time-varying term to the force



$$d(t) = D_0 + D \cos \omega t$$

Calculate directly the negative work done by friction in one period

$$\Delta E = -\oint \vec{F}\cdot dx = \oint \sigma \frac{dx}{dt}\, dx$$

$$= m\gamma \int_t^{t+T} \left(\frac{dx}{dt}\right)^2 dt$$

$$\approx m\gamma\, \omega^2 \int_t^{t+T} A^2 e^{-\gamma t} \sin^2(\omega t + \phi)\, dt$$

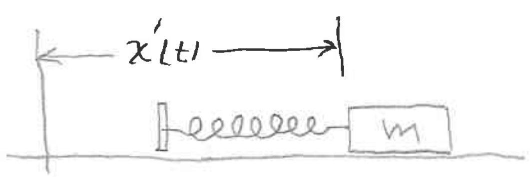
$$\approx \gamma k A^2 e^{-\gamma t} \underbrace{\int_t^{t+T} \sin^2(\omega t + \phi)\, dt}_{\frac{1}{2}\cdot T}$$

$$\Delta E = \gamma k A^2 e^{-\gamma t}\, \frac{1}{2}\, \frac{2\pi}{\omega} = 2\pi \frac{\gamma}{\omega} \cdot E \quad ✓$$

---

5. Forced simple harmonic motion

Add a harmonic, time-varying term to the force



$d(t) = D_0 + D\cos\omega t$

use $x = x' - D_0 - l$

$$m\frac{d^2x'}{dt^2} = -\sigma \frac{dx'}{dt} - k(x' - D_0 - D\cos\omega t - l)$$

equilibrium length of spring

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = kD\cos\omega t$$

Solve using complex numbers. First "simplify" the force by studying a new, second equation

$$\frac{d^2y}{dt^2} + \gamma \frac{dy}{dt} + \omega_0^2 y = \frac{kD}{m}\sin\omega t$$

Multiply 2^{nd} equation by i & add to 1^{st}

Use $z(t) = x(t) + i\, y(t)$:

$$\frac{d^2}{dt^2} z + \gamma \frac{dz}{dt} + \omega_0^2 z = \frac{kD}{m}\overbrace{(\cos\omega t + i\sin\omega t)}^{e^{i\omega t}}$$

Easily solved by $z(t) = C e^{i\omega t}$

$$-\omega^2 C e^{i\omega t} + i\gamma\omega C e^{i\omega t} + \omega_0^2 C = e^{+i\omega t}\frac{kD}{m}$$