Calculate directly the negative work done by friction in one period

$$\Delta E = -\oint \vec{F}\cdot dx = \oint \sigma \frac{dx}{dt}\, dx$$

$$= m\gamma \int_t^{t+T} \left(\frac{dx}{dt}\right)^2 dt$$

$$\approx m\gamma\, \omega^2 \int_t^{t+T} A^2 e^{-\gamma t} \sin^2(\omega t + \phi)\, dt$$

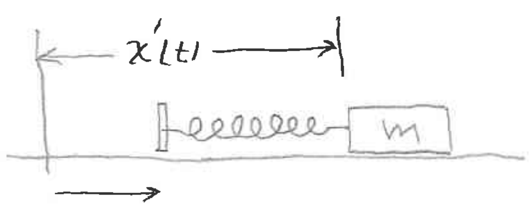
$$\approx \gamma k A^2 e^{-\gamma t} \underbrace{\int_t^{t+T} \sin^2(\omega t + \phi)\, dt}_{\frac{1}{2}\cdot T}$$

$$\Delta E = \gamma k A^2 e^{-\gamma t}\, \frac{1}{2}\, \frac{2\pi}{\omega} = 2\pi \frac{\gamma}{\omega} \cdot E \quad ✓$$

---

5. Forced simple harmonic motion

Add a harmonic, time-varying term to the force




$$d(t) = D_0 + D\cos\omega t$$





use $x = x' - D_0 - l$

$$m\frac{d^2x'}{dt^2} = -\sigma \frac{dx'}{dt} - k\left(x' - D_0 - D\cos\omega t - l\right)$$

equilibrium length of spring

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = kD\cos\omega t$$

Solve using complex numbers.

First "simplify" the force by studying a new, second equation

$$\frac{d^2y}{dt^2} + \gamma \frac{dy}{dt} + \omega_0^2 y = \frac{kD}{m}\sin\omega t$$

Multiply 2nd equation by $i$ & add to 1st

Use $z(t) = x(t) + i\,y(t)$:

$$\frac{d^2}{dt^2}z + \gamma\frac{dz}{dt} + \omega_0^2 z = \frac{kD}{m}\overbrace{(\cos\omega t + i\sin\omega t)}^{e^{i\omega t}}$$

Easily solved by $z(t) = C e^{i\omega t}$

$$-\omega^2 C e^{i\omega t} + i\gamma\omega C e^{i\omega t} + \omega_0^2 C = e^{+i\omega t}\frac{kD}{m}$$

use $x = x' - D_0 - l$

$$m\frac{d^2x'}{dt^2} = -\sigma\frac{dx'}{dt} - k(x' - D_0 - D\cos\omega t - l)^2$$

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Easily solved by: $z(t) = Ce^{i\omega t}$

$$-\omega^2 C e^{i\omega t} + i\gamma\omega C e^{+i\omega t} + \omega_0^2 C = e^{+i\omega t}\frac{kD}{m}$$



Cancel $e^{+i\omega t}$ and we have an algebraic equation for $C$:

$$C = \frac{kD/m}{-\omega^2 + i\gamma\omega + \omega_0^2}$$

Simplify by defining

$$\omega_0^2 - \omega^2 + i\gamma\omega = \sqrt{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}\; e^{i\delta}$$

$$z(t) = \frac{e^{-i\delta}\,\frac{kD}{m}}{\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}\, e^{i\omega t}$$ and

$\mathrm{re}(e^{i\omega t}e^{-\delta})$

$$x(t) = \mathrm{re}(z(t)) = \frac{kD}{m\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}\cos(\omega t - \delta)$$

solves $$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \underbrace{\frac{kD}{m}\cos(\omega t)}_{\text{inhomogenous term}}$$

what about initial conditions?

We can obtain a general solution with two arbitrary constants by adding a solution of the homogenous equation

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$$x(t) = \underbrace{\frac{kD}{m\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}}_{W(\omega)} \cos(\omega t - \delta) + \underbrace{A e^{-\frac{\gamma t}{2}} \cos\left(\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}\, t + \phi\right)}_{\text{solution of homogenous equation}}$$

where A & $\phi$ can be chosen so that $x(t)$ obeys the initial conditions

The homogeneous solution with $e^{-\gamma t/2}$ factor is transient and dies away. for large time: effect of initial condition disappears!

How do $\delta(\omega)$ & $W(\omega)$ behave as a function of $\omega$

$$W(\omega) = \frac{kD}{m\sqrt{(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2}} \qquad \tan\delta = \frac{\gamma\omega}{\omega_0^2-\omega^2}$$

$$x(t) = \frac{kD}{\underbrace{m\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}_{W(\omega)}}\cos(\omega t-\delta) + \underbrace{A e^{-\frac{\gamma t}{2}}\cos\left(\sqrt{\omega_0^2-\frac{\gamma^2}{4}}\,t+\phi\right)}_{\text{solution of homogenous equation}}$$

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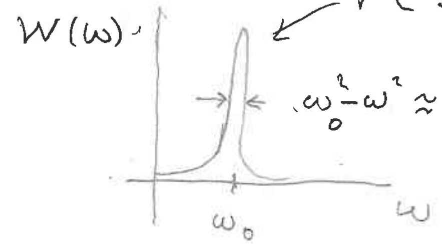
Amplitude




$$\omega_0^2-\omega^2 \approx \omega\gamma \quad \text{or} \quad (\omega_0-\omega)\overbrace{(\omega_0+\omega)}^{\sim 2\omega} \approx \gamma\omega$$

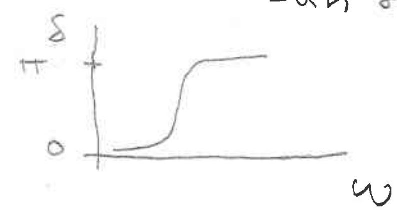
$$|\omega-\omega_0| \sim \frac{\gamma}{2}$$

for small $\gamma$

Amplitude at resonance increases by a factor $\dfrac{1/\gamma\omega_0}{1/\omega_0^2} \sim \dfrac{\omega}{\gamma} = Q$

Phase

$$\tan\delta = \frac{\gamma\omega}{\omega_0^2-\omega^2}$$



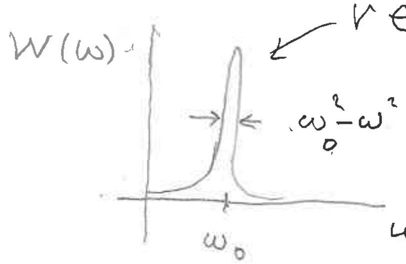

Also easy to solve for a superposition of applied forces

$$m\frac{d^2x}{dt^2} + \sigma\frac{dx}{dt} + kx = \sum_{i=1}^{N} f_i \cos\omega_i t.$$

$$x(t) = \sum_{i=1}^{N} \frac{f_i/m}{\sqrt{(\omega_0^2-\omega_i^2)^2+\gamma^2\omega_i^2}}\cos(\omega_i t-\delta_i)$$

## Amplitude

resonance




$\omega_0^2 - \omega^2 \approx \omega\gamma$ or $(\omega_0 - \omega)\overbrace{(\omega_0+\omega)}^{\sim 2\omega} \approx \gamma\omega$

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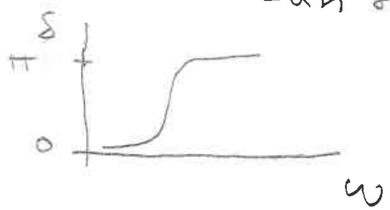


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Those terms with $\omega_i \approx \omega_0$ will contribute the largest terms to $x(t)$. The principle of violins, organs, radio receivers, etc.

---

# II Special Relativity

## A Introduction

Light travels at speed $c \approx 3\times10^{10}\ \frac{\text{cm}}{\text{sec}}$

What is this speed relative to?

1. Perhaps, like sound, relative to the medium through which it travels?

   When light moves through water

   $v = c/n \leftarrow$ index of refraction $n > 1$

   What happens if the water is moving?

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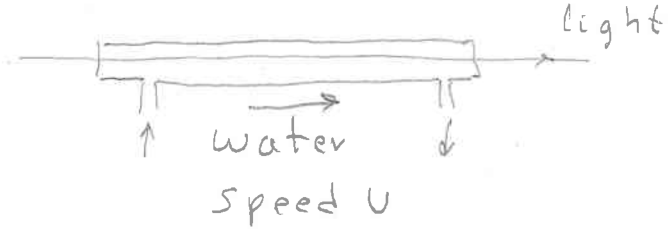
Fizeau experiment




measured speed of light is

$$v = \frac{c}{n} + u\left(1 - \frac{1}{n^2}\right) \;!$$

Of couse, expecting $\frac{c}{n} + u$ may become unreasonable as moving medium becomes dilute!

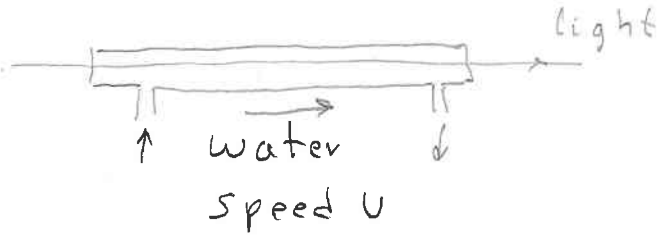
2. Perhaps c should be measured relative to the source like a bullet shot from a gun?

   No! double stars would look wierd


$$\Delta t = \frac{L}{c-v} - \frac{L}{c+v} \approx \frac{L}{c}\left(1+\frac{v}{c}\right) - \frac{L}{c}\left(1-\frac{v}{c}\right)$$

$$= \frac{L}{c} \cdot \frac{v}{c}$$

Fizeau experiment

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let $\frac{L}{c} = 1$ year & $v =$ velocity of earth

$$v = \frac{1.5\times10^{11}\,\text{m} \times 2\pi}{365 \times 24 \times 3600\ \text{sec}} = \frac{1.5\times10^{11}\,\text{m}\cdot 2\pi}{\pi\times10^{7}\,\text{sec}} \approx 3\times1$$

$$\approx 3\times10^{4}\ \frac{\text{m}}{\text{sec}}$$

$$\frac{v}{c} \approx \frac{3\times10^4}{3\times10^8} = 10^{-4}$$

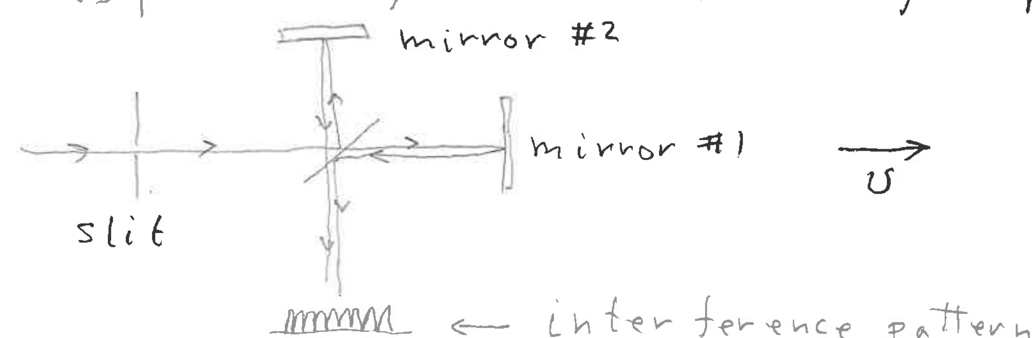
& $\Delta t = \frac{L}{c} \cdot 2\frac{v}{c} = 1\,\text{year} \times 2\times10^{-4}$

$\approx 6000$ sec

would be seen!

3. Perhaps there is a hidden medium, the "ether" through which light propagates at the speed c?

Disproven by Michelson-Morley expt.




see slit at angles where waves add

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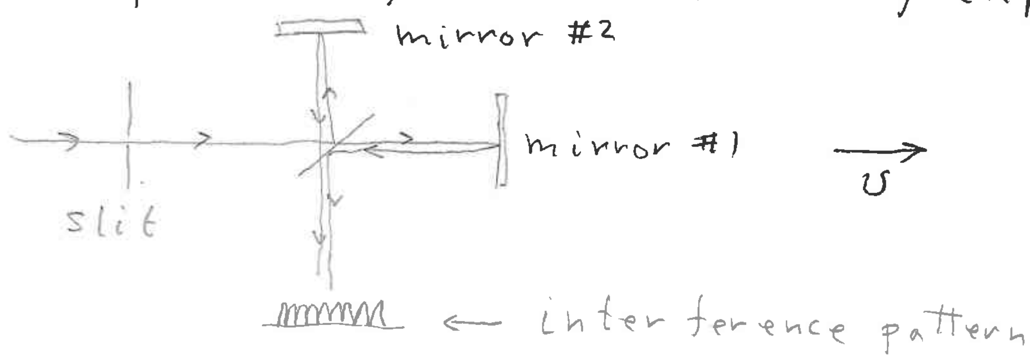



see slit at angles where waves add



Examine how the times for paths with mirror #1 & mirror #2 change if the apparatus is moving to the right through the ether with velocity $v$:

#1

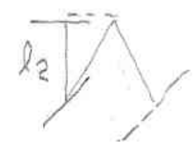

takes $t_R$ to reach mirror #1 & $t_L$ to return to half-silvered mirror

$$c\,t_R = l_1 + v\,t_R \qquad t_R = \frac{l_1}{c-v}$$

$$c\,t_L = l_1 - v\,t_L \qquad t_L = \frac{l_1}{c+v}$$

$$t_1 = t_R + t_L = \frac{l_1}{c+v} + \frac{l_1}{c-v} = \frac{2l_1}{c}\,\frac{1}{1-v^2/c^2}$$

#2




$$t_{up}\,c = \sqrt{l_2^2 + (v\,t_{up})^2}$$

$$t_{up}^2\,(c^2 - v^2) = l_2^2$$

$$t_2 = 2\,t_{up} = \frac{2l_2}{\sqrt{c^2-v^2}} = \frac{2l_2}{c}\,\frac{1}{\sqrt{1-v^2/c^2}}$$

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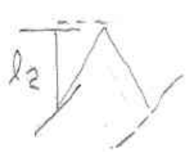

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Location of interference friges is determined by

$$t_1 - t_2 = \frac{2 l_1}{c}\,\frac{1}{1 - v^2/c^2} - \frac{2 l_2}{c}\,\frac{1}{\sqrt{1 - v^2/c^2}}$$

Rotate apparatus 90° clock wise

$$(t_1 - t_2)_{90°} = \frac{2 l_1}{c}\,\frac{1}{\sqrt{1 - v^2/c^2}} - \frac{2 l_2}{c}\,\frac{1}{1 - v^2/c^2}$$

This rotation will shift the difference between there times by

$$(t_1 - t_2) - (t_1 - t_2)_{90°} = \frac{2(l_1 + l_2)}{c}\left[\frac{1}{1 - v^2/c^2} - \frac{1}{\sqrt{1 - v^2/c^2}}\right]$$

$$\approx \frac{2(l_1 + l_2)}{c}\left[1 + v^2/c^2 - \left(1 + \frac{v^2}{2c^2}\right)\right] = \frac{l_1 + l_2}{c} \times \frac{v^2}{c^2}$$

$$\sim \frac{10\ m}{3 \times 10^8\ m/sec} \times (10^{-4})^2 = 3 \times 10^{-15}\ sec$$

↖ $v/c$ from earlier

One interference fringe ≡ one period of light oscilation $= \frac{\lambda}{c} = \frac{10^{-6} m}{3 \times 10^{-8}\ m/sec} \sim 3 \times 10^{-15}\ sec$

should see pattern move by one fringe.

It does not move!

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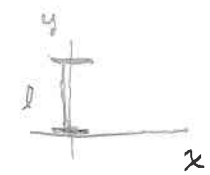
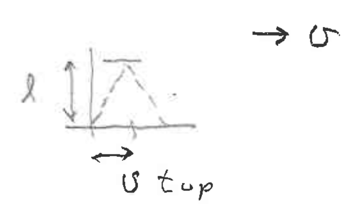
B Adopt Einstein's "principle of relativity": light moves at the speed c in all inertial reference frames. What are the implications of this assumption?

1. <u>Time</u> Design a simple clock using light propagation to measure time:




use time for light to travel up and back as standard of time

$\tau = 2l/c$. Next look at a clock moving to the right in the x direction with speed $v$




$$t_{up}\, c = \sqrt{l^2 + (v\, t_{up})^2}$$

$$t_{up}^2 (c^2 - v^2) = l^2$$

$$\text{period} = \frac{2l}{c}\frac{1}{\sqrt{1-v^2/c^2}}$$

Moving clock is slowed by factor $1/\sqrt{1-v^2/c^2}$