6. Easy to transform velocities

Look at two events in Σ' along the particles trajectory:

$$(x_1', t_1') \quad \& \quad (x_1' + u'(t_2'-t_1'),\ t_2')$$

$$x_1 = \gamma(x_1' + v t_1') \qquad x_2 = \gamma[x_1' + u'(t_2'-t_1') + v\, t_2']$$

$$t_1 = \gamma(t_1' + \tfrac{v}{c^2} x_1') \qquad t_2 = \gamma[t_2' + \tfrac{v}{c^2}(x_1' + u'(t_2'-t_1'))]$$

$$u = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\gamma[\,u'(t_2'-t_1') + v(t_2'-t_1')]}{\gamma[\,t_2'-t_1' + \frac{vu'}{c^2}(t_2'-t_1')]}$$

or $$u = \frac{u' + v}{1 + \frac{vu'}{c^2}}$$

as $u' \to c$ $\quad u \to \dfrac{c + v}{1 + \frac{v}{c}} = c$

as $v \to c$ $\quad u \to \dfrac{u' + c}{1 + \frac{u'}{c}} = c$

8. Introduce 4x4 Lorentz transformation matrices

Use coordinates (x, y, z, ct) so they all have the same units

$$x = \gamma(x' + \tfrac{v}{c}\, ct') \qquad \text{or} \qquad \begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}$$
$$ct = \gamma(ct' + \tfrac{v}{c} x')$$

$$\beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$\beta' = -\beta$ should be the inverse

$$x' = \gamma(x - \beta t) \qquad \begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}$$
$$ct' = \gamma(ct - \beta x)$$

check

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix}$$

$$= \begin{pmatrix} \gamma^2 - \beta^2\gamma^2 & 0 \\ 0 & \gamma^2 - \beta^2\gamma^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\gamma^2(1-\beta^2) = 1 \qquad \text{since} \qquad \gamma^2 = \frac{1}{1-\beta^2}$$

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$\gamma^2(1-\beta^2) = 1$ since $\gamma^2 = \dfrac{1}{1-\beta^2}$

Extend to four dimensions

$$\begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix}$$

Such a change in velocity is called a "boost".

We can combine with a rotation

$$\begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & R_{13} & 0 \\ R_{21} & R_{22} & R_{23} & 0 \\ R_{31} & R_{32} & R_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix}$$

to obtain a general Lorentz transformation matrix L

where

$$x_\mu = \sum_{\nu=1}^{4} L_{\mu\nu} x'_\nu$$

with $x_4 = ct$ and $x'_4 = ct'$

x_μ and x'_μ are components of four-vectors

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9. Invariant length

Illustrate by ignoring y & z and using two dimensions

- Rotations: consider vectors $\vec{A}$ & $\vec{B}$

$$\vec{A} = a_1 \hat{e}_1 + a_2 \hat{e}_2 = a'_1 \hat{e}'_1 + a'_2 \hat{e}'_2$$

$$\vec{B} = b_1 \hat{e}_1 + b_2 \hat{e}_2 = b'_1 \hat{e}'_1 + b'_2 \hat{e}'_2$$

use $A = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ $\quad A^t = (a_1, a_2)$

$B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ $\quad B^t = (b_1, b_2)$

then $\vec{B} \cdot \vec{A} = B^t A$

$$= \underset{1\times 2}{(b_1, b_2)} \underset{2\times 1}{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}} = \underset{1\times 1}{b_1 a_1 + b_2 a_2}$$

$$a'_i = \sum_{j=1}^{2} M_{ij} a_j \quad \& \quad b'_i = \sum_j (M^t)_{ji} b_j$$

$$\underset{2\times 1}{A'} = \underset{2\times 2}{M} \underset{2\times 1}{A} \qquad \underset{1\times 2}{B'} = \underset{1\times 2}{B} \underset{2\times 2}{M^t}$$

$$\vec{B}' \cdot \vec{A}' = (B')^t \cdot A' = \underbrace{B^t M^t}_{(B')^t} \underbrace{M A^t}_{A'}$$

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since $b_1' a_1' + b_2' a_2' = b_1 a_1 + b_2 a_2$

$$B'^t A' = B^t \underbrace{M^t M}_{\text{must } = I} A$$

$M^t M = I \Rightarrow M^t = M^{-1}$ our old result that rotation matrices are orthogonal $\equiv M^t = M^{-1}$

- <u>Boosts</u> $\quad L = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix}$ $\quad L^{-1} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}$

$$L^t \neq L^{-1}$$

Introduce a new dot product

$$(B, A)_M \equiv \sum_{i,j} B_i g_{ij} A_j = B^t G A$$

where $G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ $\quad g_{ij}$ is the metric tensor

require $(B, A) = (B', A')$

$$\underbrace{B^t}_{B'^t L^t} G \underbrace{A}_{L A'} = (B')^t G A'$$

requires $L^t G L = G$ $\quad \left\{ \begin{array}{l} \text{instead of} \\ L^t L = I \end{array} \right.$

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$G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ <u>Minkowski metric</u>

works!

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The metric g_{ij} is not changed by Lorentz transformation

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Thus, we change to a moving system the Minkowski length does not change

$$x_M^2 = (x, ct) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix} = x^2 - ct^2$$

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Try it out

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$$= x'^2\left[\gamma^2 - \frac{v^2}{c^2}\gamma^2\right] - c^2 t'^2\left[\gamma^2 - \frac{v^2}{c^2}\gamma^2\right]$$

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In four dimension

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If a and b are four vectors locating two events, then the invariant distance between x & y

$$(a-b)^2 = (x_a - x_b)^2 + (y_a - y_b)^2 + (z_a - z_b)^2 - c^2(t_a - t_b)^2$$

will be the same in all systems!

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Much more interesting than the Euclidean – three kinds of separations

1. $(a-b)^2 > 0$ a and b are space-like (time order can change)

2. $(a-b)^2 = 0$ a and b are light-like (light can go from a to b)

3. $(a-b)^2 < 0$ a and b are time-like (time order cannot be reversed

Proof (use $\vec{v} \parallel \hat{i}$)

$$\Delta t = t_a - t_b = \gamma\left(t'_a + \frac{v}{c^2}x'_a\right) - \gamma\left(t'_b + \frac{v}{c^2}x'_b\right)$$

$$= \frac{\gamma}{c}\left[c\,\Delta t' + \beta\,\Delta x'\right]$$

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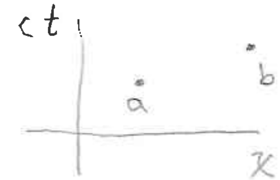
Examine

1. a & b space-like

$$0 < (a-b)^2 = (x'_a - x'_b)^2 - c^2(t'_a - b'_b)^2$$

$$\Rightarrow |x'_a - x'_b| > c|t'_a - t'_b|$$

and $\beta\Delta x'$ term can change the sign of $c\Delta t'$ term. Events a & b cannot be causally connected



$|\Delta x| > c|\Delta t| \Rightarrow$ not enough time for light to go from x_a to x_b

No information carrying signal can travel faster than light!

2. a & b light-like

$$0 = (a-b)^2 = \Delta x^2 - c^2\Delta t^2$$

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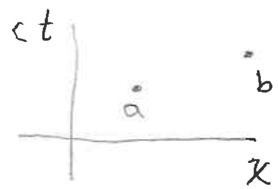

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$$0 > (a-b)^2 = (\Delta x')^2 - c^2 (\Delta t)^2$$

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enough time for a signal slower than light to reach b from a

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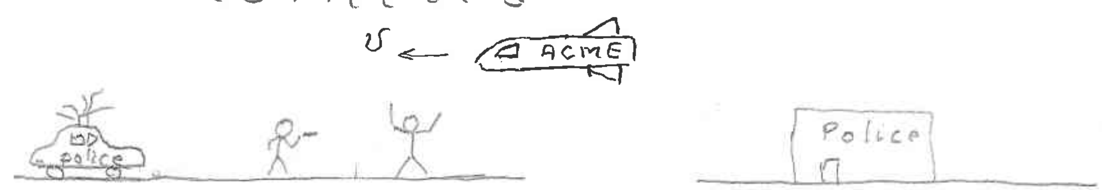

(1) ACME acausal alarm company sees robbery under way at 1 AM
(2) send superluminal signal which reaches Police before 1 AM
(3) quick acting Police send super-luminal signal to patrol car.
(4) robber arrested before crime

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10. Doppler effect for light

Σ labels these two events

$$x_1 = \gamma(x' + vt') \qquad t_1 = \gamma(t' + \frac{v}{c^2}x')$$

$$x_2 = \gamma(x' + \lambda' + vt') \qquad t_2 = \gamma(t' + \frac{v}{c^2}(x' + \lambda'))$$

$$\lambda = x_2 - x_1 + \underbrace{c(t_2 - t_1)}$$

since wave moved to the left by this amount between two measurements

$$= \gamma\lambda' + c\gamma\frac{v}{c^2}\lambda' = \frac{1}{\sqrt{1 - v^2/c^2}}\lambda'(1 + \frac{v}{c})$$

$$= \sqrt{\frac{1 + v/c}{1 - v/c}}\,\lambda'$$

wave length of light from receding star is longer, $\lambda\nu = c \Rightarrow \nu$ smaller

explains "red shift