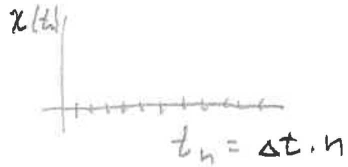
Computational diversion

Recall simple Euler integration of Newton's laws:

$$\frac{dx}{dt}(t) = v(t)\,; \quad \frac{dv}{dt}(t) = -\frac{1}{m}F(x,t)$$



$$x(t_{n+1}) \approx x(t_n) + v(t_n)\,\Delta t$$

$$v(t_{n+1}) \approx v(t_n) - \frac{F(t_n)}{m}\,\Delta t$$

Leapfrog method

$$x(t_{n+1}) = x(t_n) + \frac{dx}{dt}(t_n)\,\Delta t + \frac{1}{2}\frac{d^2x}{dt^2}(t_n)\,\Delta t^2 + O(\Delta t^3)$$

$$= x(t_n) + \frac{dx}{dt}\left(t_n + \frac{\Delta t}{2}\right)\Delta t + O(\Delta t^3)$$

since $\frac{dx}{dt}\left(t_n + \frac{\Delta t}{2}\right) \approx \frac{dx}{dt}(t_n) + \frac{d^2x}{dt^2}(t_n)\,\frac{\Delta t}{2}$

$$v\left(t_n + \frac{3\Delta t}{2}\right) = v\left(t_n + \frac{\Delta t}{2}\right) + \frac{dv}{dt}\left(t_n + \frac{\Delta t}{2}\right)\Delta t + \frac{1}{2}\frac{d^2v}{dt^2}\left(t_n + \frac{\Delta t}{2}\right)\Delta t^2$$

$$= v\left(t_n + \frac{\Delta t}{2}\right) + \frac{d^2v}{dt^2}(t_n + \Delta t)\,\Delta t + O(\Delta t^3)$$

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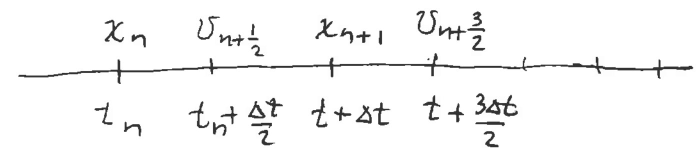
Thus, we should introduce

$x(n\Delta t)$ & $F(x(n\Delta t), t = n\Delta t)$ and $v((n+\frac{1}{2})\Delta t)$



and use

$$x(t_{n+1}) = x(t_n) + v\left(t_n + \frac{\Delta t}{2}\right)\Delta t + O(\Delta t^3)$$

$$v\left(t_{n+1} + \frac{\Delta t}{2}\right) = v\left(t_n + \frac{\Delta t}{2}\right) - \frac{F}{m}\left(x(t_n + \Delta t), t_n + \Delta t\right)\Delta t + O(\Delta t^3)$$

Now error per step is $O(\Delta t^3)$ and error after $N = \frac{T}{\Delta t}$ steps is $O(N\Delta t^3) \sim O(\Delta t^2)$

One power better than Euler with no additional work.

Return to relativity

$$p = \left(p_1, p_2, p_3, \frac{E}{c}\right)$$

$$\vec{p} = m\frac{d\vec{r}}{dt}\gamma_v\,, \quad E = mc^2\gamma_v$$

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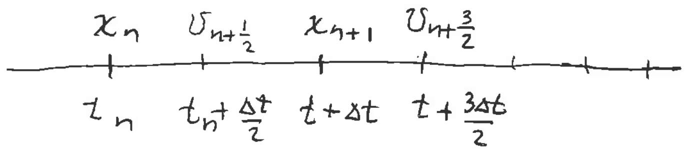

and use

$$x(t_{n+1}) = x(t_n) + v(t_n + \tfrac{\Delta t}{2})\,\Delta t + O(\Delta t^3)$$

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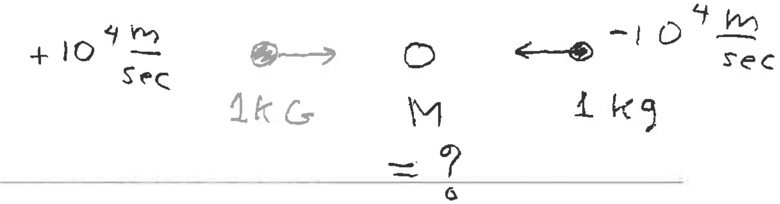


$$2\left[Mc^2 + \tfrac{1}{2}Mv^2\right] = M'c^2$$

$$M' - 2M = M\frac{v^2}{c^2} = 1\,KG \times \left[\frac{10^4}{3\times 10^8}\right]^2$$

$$\approx 10^{-9}\,KG \sim 1\,\mu g$$

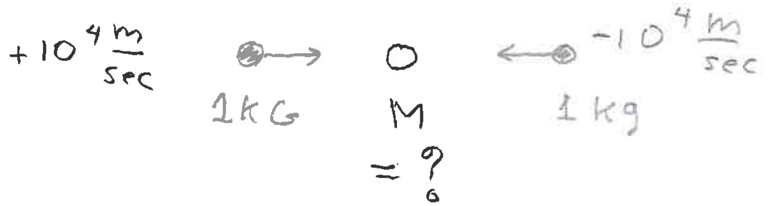
Useful relations:

If $\vec{p} = m\vec{v}\gamma_v$ & $E/c = mc\gamma_v$ are the components of a four-vector what is its invariant length?

$$\vec{p}^{\,2} - \left(\frac{E}{c}\right)^2 = m^2\vec{v}^{\,2}\gamma_v^2 - m^2c^2\gamma_v^2$$

$$= m^2c^2\left[\frac{v^2}{c^2} - 1\right]\frac{1}{1 - v^2/c^2} = -m^2c^2$$

Thus $p^2 = -m^2c^2$ and m is a scalar which was our starting hypothesis



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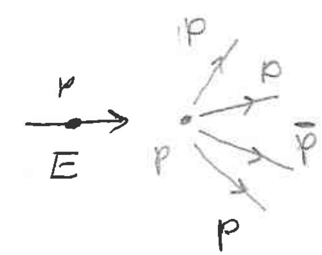
relate E & $|\vec{p}|$ without finding velocity:

$$\vec{p}^{\,2} - E^2/c^2 = -m^2c^2$$

- $\vec{p}^{\,2} = E^2/c^2 - m^2c^2$ or $p = \frac{1}{c}\sqrt{E^2 - m^2c^4}$
- $E^2/c^2 = \vec{p}^{\,2} + m^2c^2$ or $E = \sqrt{m^2c^4 + p^2c^2}$

3. Apply to the kinematics of relativistic collisions. (Easier than Newtonian because $(p_1 + p_2 + \ldots p_N)^2$ is the same in the lab & CM systems)

Problem What is the smallest energy that a proton can have in the lab to create a p-$\bar{p}$ pair when striking a target proton?



$m_p c^2 = 938$ MeV

relate E & $|\vec{p}|$ without finding velocity:

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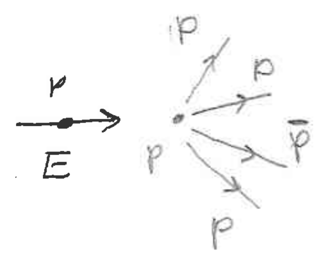

$m_p c^2 = 938$ MeV

Difficult in Lab system since beam proton must carry enough energy to allow momentum conserving recoil. Easy in CM:

In CM

$$p_b + p_t = (\vec{0},\ 4mc)$$

I Lab

beam   target at rest

$$p_b + p_t = (\vec{p}_b,\ E_b/c + mc)$$

Require $(p_b+p_t)^2_{Lab} = (p_b+p_t)^2_{CM}$

$$\vec{p}_b^{\,2} - (E_b/c + mc)^2 = 0^2 - (4mc)^2$$

$$\underbrace{\vec{p}_b^{\,2} - E_b^2/c^2}_{-m^2c^2} - 2E_b m - m^2c^2 = -16m^2c^2$$

$$E_b = \frac{1}{2m}[16-2]m^2c^2 = 7mc^2$$

Difficult in Lab system since beam proton must carry enough energy to allow momentum conserving recoil. Easy in CM:

Before

$\vec{p}_b \rightarrow \quad \leftarrow \vec{p}_t$

After

$\leftarrow$ 3 protons + 1 anti-proton at rest

In CM

$$p_b + p_t = (\vec{0},\ 4mc)$$

I Lab

beam

target at rest

$$p_b + p_t = (\vec{p}_b,\ E_b/c + mc)$$

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4. Massless particles: photons

$$E = \sqrt{p^2c^2 + m^2c^4} \rightarrow pc$$

$$|\vec{v}| = \frac{m|\vec{v}|}{m} = \frac{\gamma_v m|\vec{v}|}{\gamma_v m} = \frac{|\vec{p}|}{E/c^2} = c\,\frac{pc}{E} = c$$

as $m \to 0$, $v \to c$

Light is made up of many co-moving particles each with energy $E_{photon} = h\nu$

ν = frequency of light, $h = 6.626 \times 10^{-27}$ erg sec

$$p_{photon} = E/c = h\frac{\nu}{c} = h\frac{1}{\lambda}$$

Work out doppler effect again by Lorentz transforming 4-vectors

$\leftarrow \quad p' = \left(-p', 0, 0, \frac{E'}{c}\right)$

$E' = p'c$