4. Massless particles: photons

$$E = \sqrt{p^2c^2+m^2c^4} \to pc$$

$$|\vec{v}| = \frac{m|\vec{v}|}{m} = \frac{\gamma_v m|\vec{v}|}{\gamma_v m} = \frac{|\vec{p}|}{E/c^2} = c\,\frac{pc}{E} = c$$

as $m \to 0$, $v \to c$

Light is made up of many co-moving particles each with energy $E_{photon} = h\nu$

ν = frequency of light, $h = 6.626 \times 10^{-27}$ erg sec

$$p_{photon} = E/c = h\frac{\nu}{c} = h\frac{1}{\lambda}$$

Work out doppler effect again by Lorentz transforming 4-vectors

$\leftarrow p' = (-p', 0, 0, \frac{E'}{c})$

$E' = p'c$

$$\frac{E}{c} = \gamma\left(\frac{E'}{c} + \frac{v}{c}p'_x\right)$$

$$= \gamma\left(\frac{E'}{c} - \frac{v}{c}p'\right) \quad E'/c$$

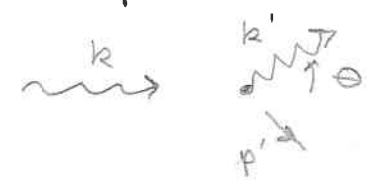
$$= \gamma\frac{E'}{c}\left(1-\frac{v}{c}\right)$$

$$\underset{\frac{c}{\lambda}}{\underset{\uparrow}{\nu}} = \gamma\nu'\left(1-\frac{v}{c}\right) = \nu'\frac{1-\frac{v}{c}}{\sqrt{1-v^2/c^2}} = \underset{\frac{c}{\lambda'}}{\underset{\uparrow}{\nu'}}\sqrt{\frac{1-v/c}{1+v/c}}$$

$$\lambda = \sqrt{\frac{1+v/c}{1-v/c}}\,\lambda'$$

Our previous result.

5. Compton scattering.



A photon scatters off a proton and comes out with a shifted ν

Before

$k = (k, 0, 0, k)$

$p = (0, 0, 0, mc)$

After

$k' = (k'\cos\theta, k'\sin\theta, 0, k')$

$p' = (p'\cos\theta', p'\sin\theta', 0, \frac{E'}{c})$

$$k + p = k' + p'$$

4-vector equation

$$\frac{E}{c} = \gamma\left(\frac{E'}{c} + \frac{v}{c} p'_x\right)$$

$$= \gamma\left(\frac{E'}{c} - \frac{v}{c} p'\right) \qquad E'/c$$

$$= \gamma \frac{E'}{c}\left(1 - \frac{v}{c}\right)$$

$$\nu = \gamma \nu' \left(1 - \frac{v}{c}\right) = \nu' \frac{1 - \frac{v}{c}}{\sqrt{1 - v^2/c^2}} = \nu' \sqrt{\frac{1 - v/c}{1 + v/c}}$$

$$\nu \uparrow \frac{c}{\lambda} \qquad \nu' \uparrow \frac{c}{\lambda'}$$

$$\lambda = \sqrt{\frac{1 + v/c}{1 - v/c}}\, \lambda'$$

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---

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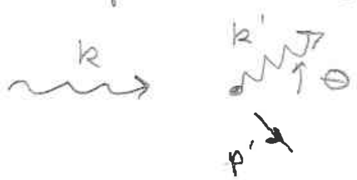

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$k = (k, 0, 0, k)$	$k' = (k'\cos\theta\; k'\sin\theta, 0, k)$
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4-vector equation

Since we do not want θ' or p'

use

$$p' = k + p - k'$$

or

$$p'^2 = k^2 + p^2 + k'^2 + 2k\cdot p - 2k\cdot k' - 2k'\cdot p$$

$$p'^2 = p^2 = -m^2c^2$$

$$k^2 = k'^2 = 0$$

$$k\cdot p = (k, 0, 0, k)\cdot(0, 0, 0, mc) = -kmc$$

$$k'\cdot p = -k'mc$$

$$k\cdot k' = (k, 0, 0, 0)\cdot(k'\cos\theta, k'\sin\theta, 0, k')$$

$$= kk'\cos\theta - kk'$$

$$0 = -\cancel{2}kmc - \cancel{2}kk'(\cos\theta - 1) + \cancel{2}k'mc$$

$$k' = \frac{kmc}{mc + k(1 - \cos\theta)}$$

$$= \frac{k}{1 + \frac{k}{mc}(1 - \cos\theta)}$$

or

$$\nu' = \frac{\nu}{1 + \frac{h\nu}{mc^2}(1 - \cos\theta)}$$

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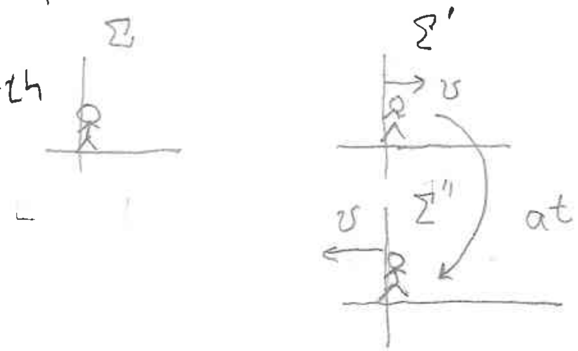
Final relativity topic (for now)

Explain twin paradox

Earth twin



moving twin

at time T' in Σ' system moving twin jumps to Σ''

Σ view

Earth twins age when moving twin start return is $\gamma T'$

Distance of moving twin is $D = \gamma T' v$

return also takes time $\gamma T'$ and

when they meet

$$(\text{age})_{E.T.} = 2T'\gamma > (\text{age})_{M.T.} = 2T'$$

Moving twin view

Let u' be Σ'' velocity with respect to Σ'

addition of velocity:

$$u' = -\frac{v+v}{1+\frac{v\cdot v}{c^2}}$$

Σ'' moves at $-v$ w.r.t. Σ

Σ moves at $-v$ w.r.t. Σ'

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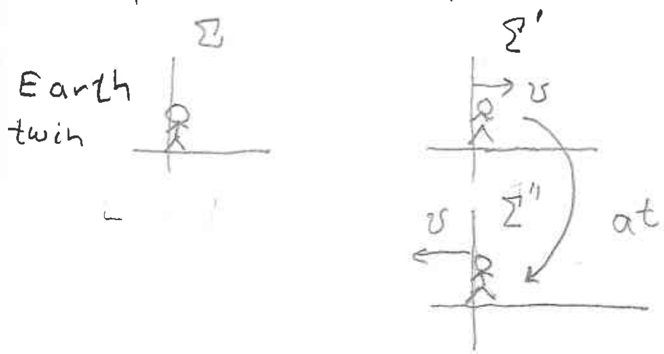

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Since it takes T' for moving twin to reach Earth, the Earth twin's age will be

$$\frac{T'}{\gamma_v} + \frac{1}{\gamma_v}\left(\gamma_u \frac{u}{c^2} v T' + T'\right)$$

$$= \frac{T'}{\gamma_v}\left[2 + \frac{1}{\sqrt{1 - \frac{4v^2/c^2}{(1+v^2/c^2)^2}}} \frac{2v^2/c^2}{1+v^2/c^2}\right]$$

$$= 2\frac{T'}{\gamma_v}\left[1 + \frac{1}{\sqrt{(1+v^2/c^2)^2 - 4v^2/c^2}} \frac{v^2}{c^2}\right]$$

$$= 2\frac{T'}{\gamma_v}\left[\frac{1 - v^2/c^2 + \frac{v^2}{c^2}}{1 - v^2/c^2}\right] = 2\gamma_v T'$$ ✓

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III Rigid body motion

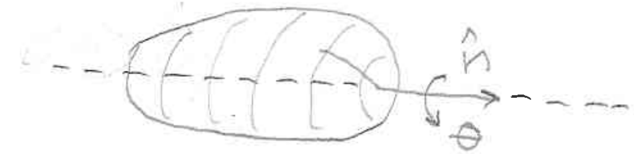
A Introduction

Since we are discussing a rigid solid we can locate it precisely using only 6 coordinates

(a) Position $\vec{R}_{cm}$ of the center of mass 3

(b) Direction of a vector $\hat{n}$ fixed in the body 2

(c) The angle the body has rotated about $\hat{n}$ from some fixed orientation 1



First study rotation about a fixed axis.

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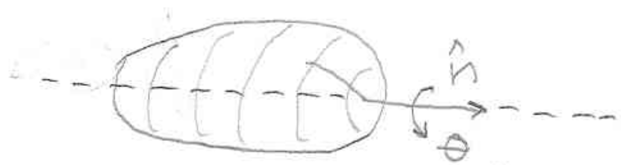


First study rotation about a fixed axis.

1. Rotation about a fixed axis A

side view   end view

Treat the solid as a sum of small masses δm_i

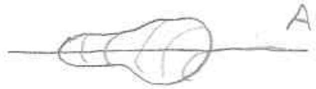
$$M = \sum_{i=1}^{N} \delta m_i$$

Each δm_i is located a distance d_i from the axis and will move in a circle of radius d_i as the body rotates about the axis A.

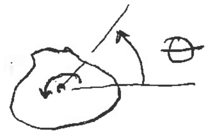
for $\dot{\theta} = \omega$

δm_i has velocity $v_i = d_i \omega$ and the total kinetic energy $= \sum_i \frac{1}{2} \delta m_i v_i^2 = \frac{1}{2} \underbrace{\sum_i \delta m_i d_i^2}_{I} \omega^2$

1. Rotation about a fixed axis A



side view



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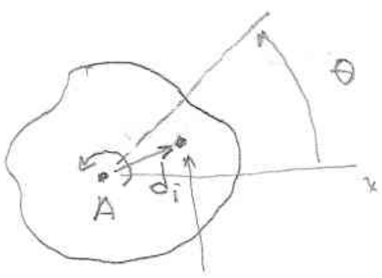

for $\dot{\theta} = \omega$

δm_i has velocity

$v_i = d_i \omega$ and

the total kinetic

$$\text{energy} = \sum_i \tfrac{1}{2} \delta m_i v_i^2 = \tfrac{1}{2} \underbrace{\sum_i \delta m_i d_i^2}_{I} \omega^2$$

Thus, $K.E. = \frac{1}{2} I_A \omega^2$ where

$I_A = \sum_i \delta m_i d_i^2$ is the moment of inertia of the body about the axis A.

Note I can be viewed as a "moment" of the mass distribution: a sum over each bit of mass δm_i weighted by d_i^2

mass density at $\vec{r}$

$$I_A = \sum_i d_i^2 \, \delta m_i = \int d^3r \, \rho(\vec{r}) \, d(\vec{r})$$

a "2nd moment" of $\rho(\vec{r})$

$$M = \sum_i \delta m_i = \int d^3r \, \rho(r)$$

the "0th moment" of $\rho(\vec{r})$

2. Use angular velocity vector and the cross product